Doppler radar and conventionality of simultaneity

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Introduction

In my essay a few days back I explained in some detail why it is impossible to determine a one-way velocity independently of the synchrony used in the descriptive framework. The theoretical reason is that one-way velocities are defined only after the synchronization is set. For someone used to abstract arguments and the rigorous application of logic that should settle the issue. But of course, there are always people that claim to have come up with a way to experimentally determine a one-way velocity "in a synchronization-independent way". Therefore, I explained by way of three concrete examples, two of them very simple, the third a bit more complex, why these attempts must fail. I demonstrated that while it is possible to set up experiments measuring only synchronization independent quantities (directly), the evaluation of a velocity from these requires the use of expressions that are different for different synchronizations. Hence, they are not synchronization independent and nor is the so-evaluated velocity.

Moreover, it is logically impossible to avoid the appearance of synchronization dependent *expressions* in a velocity formula containing only synchronization independent *terms*, because velocities *are* synchronization dependent as the existence of transformation formulas between different synchronizations demonstrates. This must show up in the velocity result somewhere. If the quantities/terms appearing in the formula are independent of synchronization, the formula itself must depend on the synchronization, meaning that there must be algebraically different formulas for different synchronizations.

Stefano Quattrini, to whom my exposition was addressed in particular, because he had put forward erroneous ideas for synchronization-independent velocity measurements in the past, now immediately contended that I made a big mistake in my first example, having used a wrong formula for the frequency change observed in a Doppler radar experiment. But there was no mistake, because his assumption was wrong that I was discussing the practical Doppler radar approach. The corresponding chapter of my treatise is titled *Velocity measurement using the Doppler effect* (not Doppler radar) and dealt with a conceptually simpler experiment than the Doppler radar approach. What I discussed was a laboratory experiment in which a light source (e.g., a monochromatic laser) is mounted on the moving object and the (anglular) frequency ω observed by the laboratory observer is the measured quantity. The emitted frequency ω' (of the laser) was assumed to have been measured before or simply to be known.

Actually, there are real-life velocity measurements that work like that even outside the laboratory. One is the determination of the radial component of the peculiar velocity of stars in our Milky Way via the Doppler shift of spectral lines. For these stars, the cosmological redshift is absent (they are too close to our position) and the gravitational redshift is either negligible in comparison with the blueshift or redshift due to the peculiar motion or it can be taken into account in the calculation, if the mass of the star is known. No radar signal is sent to these stars – the waiting time would be years and the returning signal too weak to be detectable – but the frequencies of their emission spectrum can be determined via recognition of the known patterns of spectral lines corresponding to certain atoms (for example hydrogen or helium). The emission frequencies of the spectral lines are then known, the reception frequencies are measured; the radial velocity is evaluated from this via the formula that I gave (or an equivalent one).[1]

I did mention in a later footnote that the method is used in traffic control to detect the violation of speed limits via radar Doppler shifts. Which is true. The approach *is* part of the procedure used in Doppler radar measurements. But clearly, you cannot determine the speed of cars via reception of a known-frequency radar signal coming from the car, because cars usually do not send such signals. And even if car owners were obliged by law to have a radar emitter on board so the police could measure their speeds at any time (by the experimental approach I described), there would soon be devices on the market detuning the radar signal sent by a car, so it would have a smaller frequency, misleading the receiver to believe that the car was moving at a lower velocity than it actually did.

What is done in a Doppler radar measurement is therefore different. A signal of, say, frequency ω_0 is sent by the radar source towards the target, from which it is reflected back to the receiver. The signal undergoes a Doppler shift from the point of view of the target, which receives it at a frequency ω'. The reflected signal undergoes *another* Doppler shift on return to the sender and is received there at a frequency ω. So the experiment is conceptually more complicated than the one that I discussed but as it will turn out, the resulting formulas are even (slightly) simpler. Of course, they demonstrate, once again, that no synchronization independent velocity can be extracted from the measurement. I asked Quattrini to try and derive this result himself (as this would probably enable him to understand the whys and wherefores), but I doubt he will, and so I will give the derivation myself, because it is nice referring to a practical application everyone may encounter.

Inertial systems and transformations between them

As a preliminary, I will describe the general framework of the setup and give a few formulas derived in my previous essay. We have an inertial frame S, in which the simultaneity relation leading to the definition of a global time is either obtained via Einstein synchronization, we call the corresponding time variable $\check{t}$, or via a directional Reichenbach ε-synchronization, with corresponding time t. Orienting the x axis of our frame of reference appropriately, we have the relationship ($0 < \varepsilon < 1$):

$$t = \check{t} + a_\varepsilon x \,, \qquad \text{where} \qquad a_\varepsilon = \frac{2\varepsilon - 1}{c} \,. \tag{1}$$

Moreover, our moving object will be at rest in an inertial frame S', which we will keep Einstein synchronized.[2] S moves at velocity $-v$ with respect to S', which means that S' will move at velocity v w.r.t. S, if S is Einstein synchronized, and at velocity

$$\tilde{v} = \frac{v}{1 + a_\varepsilon v} \tag{2}$$

w.r.t. S, if S is directionally ε-synchronized. Note that the fact that we have two different velocities here for a system (S') in the same state of motion is no more a contradiction than the fact that a car going on the highway at 180 km/h is in the same state of motion as the car moving at velocity *zero* in the frame of reference of another car running on the second lane at 180 km/h w.r.t. the street surface. One and the same car in a given state of motion will have

[1] Here, I refer to the formula for the case of Einstein synchronization. Astronomers do not bother with assuming other synchronizations.

[2] We could also introduce a different synchronization in S', but that would only add complexity without a corresponding gain in insight.

different velocities in different inertial frames of reference. And of course it will have different velocities with respect to different synchronies (even in the same inertial system).

In my last essay, I also gave the (generalized Lorentz) transformations between S and S' for the two simultaneity relations. When both systems are Einstein synchronized, we have

$$x' = \gamma\left(x - v\check{t}\right) , \qquad t' = \gamma\left(\check{t} - \frac{v}{c^2}x\right) , \tag{3}$$

with

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \tag{4}$$

and

$$x = \gamma\left(x' + vt'\right) , \qquad \check{t} = \gamma\left(t' + \frac{v}{c^2}x'\right) . \tag{5}$$

When S is directionally Reichenbach ε-synchronized (along the x direction), the corresponding relations are:

$$x' = \frac{\gamma}{1 - a_\varepsilon\tilde{v}}(x - \tilde{v}t) , \qquad t' = \gamma\left(t - b_\varepsilon x\right) , \quad b_\varepsilon = a_\varepsilon + \frac{\tilde{v}/c^2}{1 - a_\varepsilon\tilde{v}} \tag{6}$$

with

$$\gamma = \left(1 - \frac{\tilde{v}^2/c^2}{(1 - a_\varepsilon\tilde{v})^2}\right)^{-1/2} \tag{7}$$

and

$$x = \gamma\left(x' + \frac{\tilde{v}}{1 - a_\varepsilon\tilde{v}}t'\right) , \qquad t = \gamma\left(\frac{t'}{1 - a_\varepsilon\tilde{v}} + b_\varepsilon x'\right) . \tag{8}$$

It is seen immediately that for $\varepsilon = 1/2$, i.e., $a_\varepsilon = 0$, these equations reduce to the standard Lorentz transformations with $\tilde{v} = v$. The inversion of formula (2) is

$$v = \frac{\tilde{v}}{1 - a_\varepsilon\tilde{v}} . \tag{9}$$

Analysis of the Doppler radar experiment

We wish to derive a relationship between the velocity (in either synchronization) and the two measured frequencies ω_0 and ω. We can do so by first obtaining the Doppler effect for the radar signal from the emitter to the reflecting target, i.e., the relationship between ω_0 and ω', and then the Doppler effect for the reflected signal moving from the target back to the receiver (that is next to the emitter) in the laboratory system (S), i.e., the relationship between ω' and ω. Since the second calculation has already been done in *Conventionality of simultaneity and velocity measurement* (CSVM), we may copy the result from there and do only the first calculation here.

The target is assumed to move along the x axis to the right, and the ultimate goal is to determine its velocity from the Doppler effect results.

As in CSVM, we exploit that the phase of the light wave is a relativistic invariant. The first wave is outgoing and moving to the right (towards positive x), so its phase is describable by

$$A(x, \check{t}) \propto \mathrm{e}^{\mathrm{i}(\omega_0\check{t} - \check{k}_0 x)} , \tag{10}$$

and we may set

$$\omega_0 \check{t} - \check{k}_0 x = \omega' t' - k' x' \,. \tag{11}$$

Replacing x' and t' on the right-hand side with their expressions given by the Lorentz transformation (3), we obtain an equation that must be identically satisfied in $\check{t}$ and x, yielding expressions for ω_0 and $\check{k}_0$ in terms of ω' and k'. These read

$$\omega_0 = \gamma \left(\omega' + v k'\right) \,, \qquad \check{k}_0 = \gamma \left(k' + \frac{v}{c^2}\omega'\right) \,. \tag{12}$$

Inserting the dispersion relation $\omega' = ck'$ that holds in S', we find

$$\omega_0 = \gamma \left(1 + \frac{v}{c}\right)\omega' = \left(\frac{1+v/c}{1-v/c}\right)^{1/2} \omega' \,, \qquad \check{k}_0 = \gamma \left(1 + \frac{v}{c}\right) k' = \left(\frac{1+v/c}{1-v/c}\right)^{1/2} k' \,. \tag{13}$$

In CSVM, we obtained the relationship between ω' and ω (Eq. (17) there):

$$\omega = \left(\frac{1-v/c}{1+v/c}\right)^{1/2} \omega' \,. \tag{14}$$

Using $\omega' = \left(\frac{1-v/c}{1+v/c}\right)^{1/2} \omega_0$, which follows directly from Eq. (13), we finally get

$$\omega = \frac{1-v/c}{1+v/c}\,\omega_0 \,, \tag{15}$$

and this can be easily solved for v:

$$\boxed{v = c\,\frac{1-\omega/\omega_0}{1+\omega/\omega_0}} \,. \tag{16}$$

Interestingly, the result Eq. (15), which is a fully relativistic formula, is also obtained for a non-relativistic Doppler experiment, say, a sonar experiment instead of a radar experiment (of course, the speed of light has to be replaced by the speed of sound then).[3]

Eq. (16) was derived assuming Einstein synchronization in S (where we wanted to determine the velocity). So the result is not likely to be independent of synchronization, a question that we will explore further now.

To do so, we derive the Doppler effect formula for a time variable t that is Reichenbach synchronized according to Eq. (1). Of course, the phase of a light wave is still observer

[3]This is due to the combination of two signals moving in opposite directions. Assume ν_0 to be the frequency of a sound signal sent towards the target. Consider two wave crests following each other at a distance λ_0, the wavelength. Let the first hit the target at x_1; the second then is at $x_1 - \lambda_0$, it races towards the target at speed c_s, and the target moves (away) at speed v; the time interval $\Delta t'$ until arrival of the second crest is given by $x_1 + v\Delta t' = x_1 - \lambda_0 + c_s \Delta t'$. We get $\lambda_0 = (c_s - v)\Delta t'$. The time interval between the sending of successive crests is $\Delta t = 1/\nu_0$, implying $\lambda_0 = c_s \Delta t = c_s/\nu_0$. The frequency $\nu' = 1/\Delta t'$ of the sound signal hitting the target then is given by $c_s/\nu_0 = (c_s - v)/\nu'$, i.e. $\nu' = (1 - v/c_s)\nu_0$, which is the formula for the non-relativistic Doppler effect experienced by a moving receiver. Consider now two successive crests of the reflected wave, the first emitted at x_2. During the time interval $\Delta t'$ until emission of the second crest, the emitted crest moves to $x_2 - c_s\Delta t'$, whereas the target moves to $x_2 + v\Delta t'$. So the distance between the two crests is $\lambda = x_2 + v\Delta t' - x_2 + c_s\Delta t' = (c_s + v)\Delta t'$. The velocity of the wave is c_s to the left. Its frequency can then be obtained from $\lambda = c_s/\nu = (c_s + v)/\nu'$. Hence, $\nu = \nu'/(1 + v/c_s)$. This is the formula for the Doppler effect due to a moving emitter. The combination of the two formulas gives $\nu = \nu_0(1 - v/c_s)/(1 + v/c_s)$, which is formally the same as the relativistic result. The difference is that in the relativistic case the formulas for the Doppler effects with moving receiver and moving emitter are symmetrical, whereas they are substantially different when the wave speed is determined by an underlying medium.

invariant, so we can equate the phase in S' and that in the now Reichenbach synchronized system S:

$$\omega_0 t - k_0 x = \omega' t' - k' x' \,. \tag{17}$$

where in writing ω_0 we employ that the frequency measured at a fixed position in S (i.e., with one clock) is independent of which of the two synchronizations under consideration we use. However, we write k_0 for the wave number instead of $\check{k}_0$, because we cannot expect the wavelength to remain unchanged. (The wavelength is the distance between two successive maxima of the oscillation at a fixed time. But the meaning of "fixed time" at the – different – positions of the maxima is synchronization dependent.)

Using (6) in (17), we are able to derive the Doppler effect formula. We have

$$\omega_0 t - k_0 x = \omega' \gamma (t - b_\varepsilon x) - k' \frac{\gamma}{1 - a_\varepsilon \tilde{v}} (x - \tilde{v} t) \tag{18}$$

and collecting coefficients of t and x, we arrive at

$$\omega_0 = \gamma \left(\omega' + \frac{\tilde{v}}{1 - a_\varepsilon \tilde{v}} k' \right) \,, \tag{19}$$

$$k_0 = \gamma \left(\frac{k'}{1 - a_\varepsilon \tilde{v}} + b_\varepsilon \omega' \right) \,. \tag{20}$$

Using $\omega' = ck'$ again, we find for the frequency

$$\omega_0 = \gamma \omega' \left(1 + \frac{\tilde{v}/c}{1 - a_\varepsilon \tilde{v}} \right) \tag{21}$$

which may be solved for ω'

$$\omega' = \frac{\omega_0}{\gamma} \left(1 + \frac{\tilde{v}/c}{1 - a_\varepsilon \tilde{v}} \right)^{-1} \,. \tag{22}$$

The relationship between ω and ω' has been calculated in CSVM, from where we copy Eq. (30)

$$\omega = \gamma \omega' \left(1 - \frac{\tilde{v}/c}{1 - a_\varepsilon \tilde{v}} \right) \,, \tag{23}$$

which, in combination with Eq. (22) yields

$$\omega = \omega_0 \frac{1 - (\tilde{v}/c)/(1 - a_\varepsilon \tilde{v})}{1 + (\tilde{v}/c)/(1 - a_\varepsilon \tilde{v})} = \omega_0 \frac{1 - a_\varepsilon \tilde{v} - \tilde{v}/c}{1 - a_\varepsilon \tilde{v} + \tilde{v}/c} \tag{24}$$

This can be easily solved for $\tilde{v}$:

$$\begin{aligned}
\omega - a_\varepsilon \omega \tilde{v} + \omega \frac{\tilde{v}}{c} &= \omega_0 - a_\varepsilon \omega_0 \tilde{v} - \omega_0 \frac{\tilde{v}}{c} \\
\tilde{v} \left(\frac{\omega}{c} - a_\varepsilon \omega + \frac{\omega_0}{c} + a_\varepsilon \omega_0 \right) &= \omega_0 - \omega \\
\tilde{v} \left(1 + \frac{c a_\varepsilon (\omega_0 - \omega)}{\omega_0 + \omega} \right) &= c \frac{\omega_0 - \omega}{\omega_0 + \omega} \\
&\boxed{\tilde{v} = c \frac{1 - \omega/\omega_0}{1 + \omega/\omega_0} \frac{1}{1 + a_\varepsilon c \frac{1 - \omega/\omega_0}{1 + \omega/\omega_0}}} \,.
\end{aligned} \tag{25}$$

We obtain, as in similar instances before, a different formula for the velocity within a directionally Reichenbach ε-synchronized system than for measurement of the *same* quantities ω, ω_0 in an Einstein synchronized system. It may be noted that the relationships Eq. (25) and Eq. (16) satisfy $\tilde{v} = v/(1 + a_\varepsilon v)$, i.e., Eq. (2). This is a useful check, demonstrating the consistency of the calculations.

The velocities $\tilde{v}$ and v correspond to the same state of motion, the former for system S with directional Reichenbach synchronization, the latter for S with Einstein synchronization. This is no more surprising than the fact that a car can have two different velocities in two different frames of reference. The velocities and the kinetic energies of the car described may be very different in the two frames, although it is nothing but the same car in the same state of motion. Velocity differences due to different synchronizations are much smaller and lead to much less drematically different descriptions for everyday situations. If the car goes at 180 km/h in an Einstein synchronized frame, then $|a_\varepsilon|\, v$ does not exceed 180 km/h/c = 50 m/s/(3×10^8 m/s) = 1.67×10^{-7}, hence the car velocity in a Reichenbach synchronized frame will be between 180 km/h$\times(1 - 1.67 \times 10^{-7})$ and 180 km/h$\times(1 + 1.67 \times 10^{-7})$, which is essentially indistinguishable from the Einstein value. The situation changes, of course, for velocities near the speed of light.

To sum up, if the same experiment is done twice in the same system S but with the two different synchronizations,[4] the measured frequencies ω and ω_0 will be the same in both instances, but the results will be considered measurements of different velocities. The reason is that the actual quantity measured is not a velocity but two frequencies and that theoretical reasoning is required to turn the frequency result into a velocity measurement. Therefore, even though frequencies can be measured at one position with the help of a single clock, we *cannot* use this to determine a one-way velocity *independently of synchronization*.

[4] In fact, we do not even have to do the experiment twice. A single experiment is sufficient, if we imagine to have *two* clocks at any point of space, one showing the Einstein time $\check{t}$ and the other the Reichenbach time t. The clocks need not all be real, because in the experiment we need only one of them, and we may conveniently put the origin of our coordinate system at its position, so it will measure both times at once, because their offset is zero at the origin. The velocity obtained from the experiment will then depend on which of the two times we use for its definition. Sorry, if that sounds trivial. That is, because it *is* trivial.