The identically accelerated twins revisited

Klaus Kassner 2 April 2018

About two weeks ago, Stefano Quattrini published a one-page exposé "Breaking the Lorentz invariance", in which he claims to demonstrate Lorentz invariance to be broken in a variant of the scenario of the accelerating twins discussed by Boughn [1]. Quattrini's analysis is incorrect and the reason is the same as with so many other insufficient discussions of special relativistic paradoxes: lack of appreciation of the relativity of simultaneity and its consequences.

There is not only one error. Quattrini uses a formula for the time difference between the age of the two twins that is valid only, if their acceleration is stopped at the same time in the inertial system, in which they originally were at rest. His stopping procedure will not achieve this, hence the formula for the time difference used by him is wrong. But this is only a minor objection.

The main issue is that with his stopping procedure the minimum stopping time is not determined by the leading twin but by the trailing one and therefore the paradox dissolves. The same is true if an appropriate stopping procedure different from Quattrini's is devised that makes Boughn's formula exact.

Moreover, if stopping is made at *one* time in either of *the twins' frames*,[1] then the twins will not move at the same velocity after the switch-off of their engines. It is even possible (and no contradiction at all) that the trailing twin never starts his journey in this case.

Minkowski diagrams

Before considering the proposed scenario, I would like to first revisit Boughn's identically accelerated twins using Minkowski diagrams.

It is obvious that the Boughn paper has not been understood by many. To some extent this may be due to the fact that he uses language that may confuse rather than clarify, even though technically, he does not commit any major error. When I first read the Boughn paper, I was ready to write a comment, pointing out this slight abuse of language. But then I saw the paper had appeared in 1989 already, more than 20 years before I read it, so it was probably pointless to comment on it. In fact, a comment had been published two years after [2], and a very nice exposition of the kind of different narratives that special relativity implies for different observers was given in 1996 by Price and Gruber [3] (who have teamed up for other useful articles on relativity [4] in the Americal Journal of Physics).

The problem with Boughn's language is the following: he keeps talking about "identically accelerated twins" without emphasizing *in which frame the twins are identically accelerated*. Now it turns out that in the frame, where the twins are identically accelerated (the parents' frame) they age at the same rate, whereas in the frame(s), in which they do not age at the same rate, they are not identically accelerated. Stated like this, Boughn's result might have looked much less surprising, and maybe that is the reason for his avoiding to point out that in the final frame the twins were not identically accelerated.[2] Or else he simply did not see it.

In any case, a nice qualitative discussion of the situtation with onla very few formulas (and simple ones) is possible using Minkowski diagrams. I will first give an explanation of the

[1] I will discuss below how to achieve this.

[2] Contrary to Newtonian mechanics, acceleration is not invariant under a change of inertial frame.

original problem described by Boughn and then continue with a consideration of the Quattrini argument. In Fig. 1, the $x\,ct$ coordinate system is the frame P of the parents who stay at rest in an inertial system. The blue curves are the world lines of the twins, Dick (left) and Jane (right), who start to get accelerated at $t = 0$, accelerate until time t_2 and then switch off their engines, coasting at the same velocity afterwards. As usual in relativity (and for a reason!), the time axis is actually taken to be labeled by ct. World lines of massive objects must then have a slope exceeding one, and the slope is the inverse of the velocity, measured in units of c, i.e., v/c. So Dick's and Jane's world lines become straight lines after t_2 again, but now are tilted at an angle φ satisfying $\tan\varphi = c/v$, v being their final velocity.

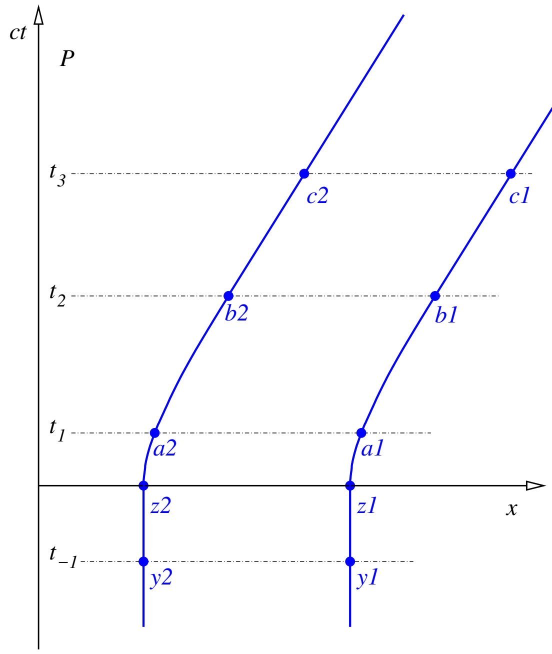

Fig. 1: World lines of the identically accelerating twins.

Identical acceleration of the twins means that the two world lines are the same curves up to a horizontal shift. Since the proper time of each twin is given by an integral $\tau - \tau_0 = \int_{t_0}^{t} \sqrt{1 - \frac{v^2}{c^2}}\mathrm{d}t'$ and v is the same for both twins at the same time t, it is clear that the twins have the same age at certain corresponding events. For example, Jane has the same age at event $a1$ on the figure as Dick at event $a2$, and Jane has the same age at event $c1$ as Dick at event $c2$, etc. For all drawn pairs of events, Jane has the same age at event $\ldots 1$ as Dick at event $\ldots 2$. And this is a relativistically invariant statement! In the frame P, events $a1$ and $a2$ are at the same time, as are $b1$ and $b2$, etc. This is the meaning of the statement that the twins age at the same rate in the parents' frame. They do not necessarily age at the same rate in other inertial frames, as these do not have to agree that, say, $a1$ and $a2$ happen at the same time. Hence, Jane and Dick would have the same age at *different* times in such a frame, meaning that they cannot have the same age at the *same* time.

We can further embellish the diagram by including the frames of Dick and Jane at some points. In general, we consider as frame of either twin a locally comoving inertial frame. As I will discuss shortly, such a frame can be constructed easily by purely graphical means. This is done in the next picture, Fig. 2.

The comoving frame of Dick is called B during his acceleration phase and C in the final state, wheras A is a comoving frame of Jane during her acceleration phase and C' is the final inertial system of Jane. We shall see later that C and C' can be identified with a common inertial system of Dick and Jane late enough into the course of the scenario, while B and A cannot.

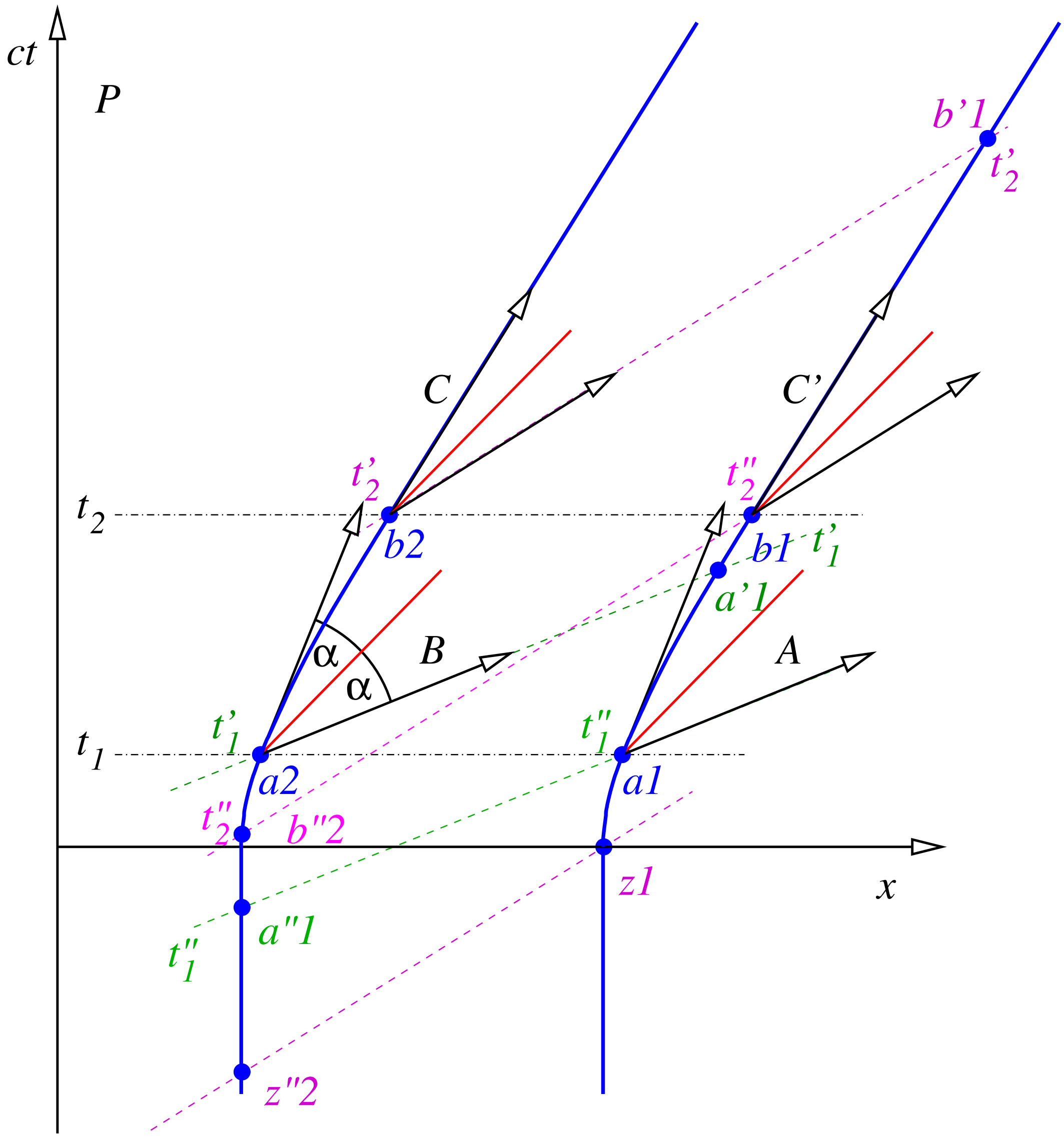

Fig. 2: Minkowski diagram for the identically accelerating twins.

Coordinate systems describing the local frames have been drawn along each of the world lines. How do we get them without calculation? The time axis is easy. It must be tangent to the world line. Next we can draw the world line of a light ray sent from the origin of the local coordinate system in the x direction. It must have slope 1 (i.e., 45 degrees) in the $x\,ct$ diagram, because light sent under these conditions simply satisfies $x = x_0 + c(t - t_0)$. World lines of light sent along the x or $-x$ direction are always parallel to the first or the second bisector in a Minkowski diagram. But in the local frames, light moves at speed c, too. And if their time axes are taken to be c times a time coordinate, a light ray must follow a bisector in the local frame as well. But that means that we can very easily construct the x axis of the local frame, just by doubling the angle between the time axis and the world line of light (this is explicitly shown in Fig. 2 for one of the angles, called α).

The fact that the x axes of the local frames are not parallel to the x axis of the frame P straightforwardly expresses the relativity of simultaneity. In each of the local frames, lines of equal time are given by parallels to their x axis. Since the local x axis is tilted with respect to the x axis in P, it is clear that we do not have the same notions of simultaneity. For example, the event $a2$ given by the intersection of the line $t = t_1$ with Dick's world line is simultaneous with the event $a1$ given by the intersection of the same line with Jane's world line in the parents' frame. As we have argued before, both twins have aged by the same amount of time at these two events, meaning they have the same age at the same time, $t = t_1$, according to the parents. However, the same event $a2$ is, *in Dick's frame* (who reads the time $t' = t'_1$ off his clock) simultaneous with the intersection $a'1$ of the prolongation of Dick's x axis (displaying coordinate x') with Jane's world line. These two events have the same time $t' = t'_1$ according to Dick. But at the point $a'1$ of Jane's world line, her velocity is higher than Dick's (because its slope is smaller than that of Dick's world line at the origin of his x axis). So she must move away from him. Moreover, she will not agree that this event is simultaneous with the time t_1 of their parents. Rather, she finds that the intersection of $t = t_1$ with her world line is simultaneous with a point at her time $t'' = t''_1$ on Dick's world line, given by the backward prolongation of her x axis to its intersection with Dick's world line. In our drawing, that time was even before Dick started to accelerate![3] Since the x axis of Jane's comoving inertial system will, at all times of her acceleration period, intersect with Dick's world line at points where the latter has a larger slope than Jane's world line at the point, from which the x axis was drawn, Jane will find that Dick is always *slower* than she herself. So she will agree with Dick that their distance is increasing.

Another thing that can be immediately seen from the diagram is that the stopping time t_2 corresponds to a time t'_2 for Dick, at which he finds that Jane has already stopped quite some time ago, because she is "right now" at the event $b'1$ corresponding to his time t'_2 on her world line. On the other hand, Jane will, when she is stopping, at her time t''_2, consider Dick still being in his acceleration phase (at $b''2$, corresponding to her time t''_2 on his world line). Note that numerically $t'_2 = t''_2$ – these are times measured in different local frames.[4] To obtain the age difference some time after the switching-off of their engines, we need the time differences as measured by one of the observers in his or her local frame. Only if they are at rest with respect to each other, will they agree on their age difference. But this is the case for Dick after event $b2$ and for Jane after event $b'1$. So, one way to obtain the age difference is to calculate the time difference between the events $b'1$ and $b1$ in frame C, referring to the known times and positions in the parents' frame.

Let us denote by x_2 Dick's position at event $b2$, in the frame P. He is sitting at $x' = 0$ by definition and his time is t'_2. Then, the Lorentz transformation between frames C and P reads $\left(\gamma = 1/\sqrt{1 - \frac{v^2}{c^2}}\right)$:

$$\begin{aligned} t' - t'_2 &= \gamma \left(t - t_2 - \frac{v}{c^2}(x - x_2) \right) , \\ x' &= \gamma \left(x - x_2 - v(t - t_2) \right) . \end{aligned} \tag{1}$$

At $t = t_2$ and $x = x_2$, we obviously have $t' = t'_2$ and $x' = 0$. But we can also calculate the time and position, in Dick's frame, of the event $t = t_2$ and $x = x_2 + H$, corresponding to Jane's switching off her engine. We find

$$\begin{aligned} t' - t'_2 = \tilde{t}'_2 - t'_2 &= \gamma \left(t - t_2 - \frac{v}{c^2}(x - x_2) \right) = -\gamma \frac{vH}{c^2} , \\ x' &= \gamma H . \end{aligned} \tag{2}$$

[3] I will come back to that point later, as it may confuse people.

[4] And of course, $t'_1 = t''_1$.

So the distance in frame C has increased from the original distance H to γH, and the age difference of the twins will be $t'_2 - \tilde{t}'_2 = \gamma \frac{vH}{c^2}$. Note that it is Jane who is older, because the event $b1$ that happens at time $\tilde{t}'_2$ in Dick's frame, happens at time $t''_2 = t'_2 > \tilde{t}'_2$ in her frame, as the times at which both twins switch off their engines are the same in their respective local frames. So Jane's age at t'_2 (in Dick's frame) is by $t'_2 - \tilde{t}'_2$ larger than t'_2. Note that the twins agree on their age difference, as soon as agree on both their engines being switched off, which is above the x axis of C at the intersection point with the t_2 line ($b2$) . Below that line, Jane will not consider C' and B (which corresponds to C before event b2) to move at the same speed. Later, the C and C' inertial systems can be identified, their only difference being a constant offset in temporal and spatial coordinates.

But this also shows that earlier on the world lines, the inertial systems B and A can *never* be identified (except at the time when they have velocity zero). For Dick always will find that Jane has a *larger* velocity at any time in his comoving inertial system than he himself (the intersection point of his x axis is always at a later proper time on Jane's world line than on his). Jane will always find that Dick has a *smaller* velocity than she herself (the intersection point of her x axis is always at an earlier proper time on Dick's world line than on hers). Since their velocities are different, they never share a common inertial system in which they would be both at rest, as long as one of them is accelerating. Only before they have started their acceleration period (and agree on neither of them having started yet) and after they have stopped it (and agree on both having stopped) can they be at rest in a common inertial system. Clearly, while they accelerate, both will find that the other is accelerating at a different rate, so they are not identically accelerating!

It should also be noted that the time dilation that arises here is not quite of the same kind as the time dilation typically discussed when comparing observers in inertial frames of reference moving with respect to each other. There we have *mutual* time dilation, i.e., each of the observers finds the other to have a slower clock rate than they have themselves, and the effect depends on v^2/c^2. Here, the situation is different. We can read off the diagram that the reason for the faster aging of Jane is simply that the x axis of Dick turns upward and gets steeper and steeper as he accelerates. This means that points that are simultaneous with his clock ticks move upward on the world line of Jane, so correspond to later times on her clock. This effect is essentially *linear* in v/c.[5] Since Dick is behind Jane, her x axis turns downward with increasing velocity,[6] hitting earlier times on his world line than on hers. So he ages more slowly than she does. In fact, it is possible for him to age backwards, as the picture shows, if only their initial distance is large enough. Jane's simultaneity line t''_1 hits Dick's world line at event $a''1$ while he is still at rest, even though they both agreed that they started at $t = t' = t'' = 0$.

While this may be confusing, it is not a problem for the theory. Simultaneity at a distance is largely a matter of definition, the only requirement being that events that are declared simultaneous must be spacelike. They cannot be timelike or null. What we discussed, was simultaneity obtained by Einstein synchronization in inertial frames. There is no way of operationally verifying this simultaneity in flight. All Dick and Jane can do is exchange signals. And any sequence of signals sent by either of them will be received by the other in the same order as they were sent, as long as the signals do not go faster than light. This is exemplified in Fig. 3 where I have drawn the world lines of the two of them and a number of light signals sent in both directions.[7] The signals are sent at proper times τ_1, τ_2, etc. by Dick, and their arrival events, simply denoted by 1, 2, etc. on Jane's world line, are in the

[5] It also contains higher-order contributions in v/c. But the leading-order term is linear and not quadratic.

[6] He is at negative x values for her, while she is at positive x values for him.

[7] I have moved their world lines a little closer together to make sure all the signal receptions fit into the image.

same order as the emission events, so there is no observable sign of anything unusual in Dick's course of time. The same holds, mutatis mutandis, for signals being sent from Jane to Dick.

So the only way Jane can state that Dick's time is going backward for some time around the start is *by theory*. She can calculate his proper time in terms of the coordinate time in her comoving inertial system and find that during some interval the coordinate time is running backwards in comparison with the proper time. But this is not a problem, all it shows is that the coordinate time of her comoving inertial system is not well-suited for an intuitive description of what happens at Dick's position. Comoving inertial systems are useful only locally when there is acceleration, and Jane's coordinates are not useful in describing Dick's life, if he is too far from her. Of course, for him there is nothing particular happening while Jane's coordinate time runs in the opposite direction of his proper time at his position.[8] Einstein synchronization is very useful for a description of temporal sequences in inertial systems, but is loses some of this utility in accelerating systems such as the rotating disk [5].

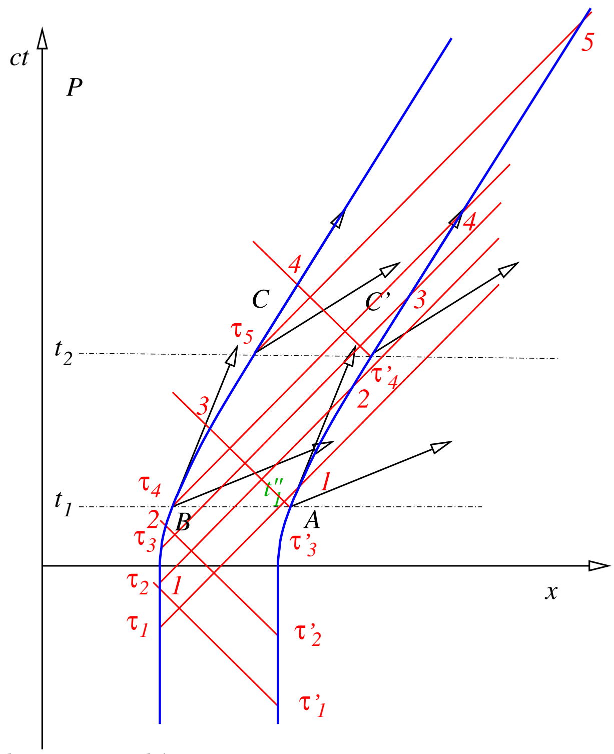

Fig. 3: Signal exchange. Dick's proper times at emission are τ_n, $n = 1 \ldots 5$, arrival of the signal is only denoted by n. Jane's proper times at emission are τ'_n, $n = 1 \ldots 4$, arrival is again only denoted by n.

Finally, it is helpful to consider the whole story from the point of view of the final inertial frame, when C and C' may be identified. Lines of simultaneity in this frame are drawn in dark magenta in Fig. 2. The line connecting the events $z1$ and $z''2$ demonstrates that in this frame, Jane starts her journey before Dick. She also finishes it before him (at the intersection $b1$ of the t''_2 line with Jane's world line, Dick still being at $b''2$). Since in this frame both twins are moving initially (towards the negative x direction) and are at rest finally, this means that Dick keeps moving while Jane is getting slower and hence he ages more slowly than she. None of the lines of simultaneity in frame CC' intersects the two world lines at the same velocity between events $z1$ and $b'1$ (exclusively). As long as one of the intersection points lies in the acceleration

[8]Closer to Jane's position, her coordinate time will be directed to the future of all local proper times.

part of the world lines, Dick is always moving faster towards the negative x direction. So the two twins are certainly *not identically accelerated* in the final frame of reference. And the fact that Dick is younger is easily explained by his motion at a speed that never falls below that of his sister throughout the time intervals considered.

Note also that the age difference is a "difference at a distance". Dick and Jane cannot compare clocks or calendars directly, they have to communicate via signals. What will be their age difference, if they come together to compare clocks? That depends on *how* precisely they get together. If they do it by slow enough motion in the final frame, their age difference will remain the one calculated above. If Dick keeps his velocity unchanged and Jane moves towards him sufficently fast, she will age more slowly during that time and they will find a smaller age difference, she may even be younger than he on meeting. If Jane stays at a fixed position in the final frame and Dick joins her, the age difference will at least be the amount calculated, because he can only age more slowly than her during the trip. Finally, if they return to their parents' frame, each by exactly the same acceleration program – *as seen in* P, they will have the same age again.[9]

Involving another moving frame

Having discussed the Boughn scenario, let us now have a look at Quattrini's changes. The first thing is that he puts a comoving system S between the two. Obviously, he thinks that if S instantaneously moves at the same speed as B and A that they are essentially the same inertial system. As we have noted, things are a bit more complicated than that, see Fig. 4.

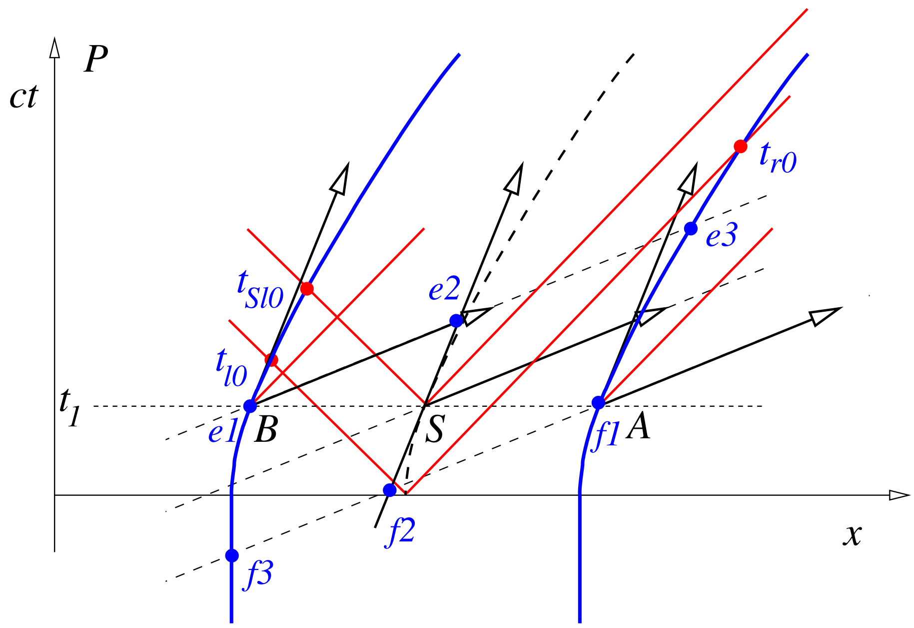

Fig. 4: Dick's and Jane's frames B and A together with an instantaneously comoving inertial system S, i.e., S does not move along the dashed world line but along a tangent to it.

First, we have to decide whether S itself is accelerated in parallel with Dick and Jane. If this is the case, neither Dick nor Jane will ever be at rest with respect to S in an inertial system comoving with S (except at $t = 0$). So this is not a useful construction.

[9]This means their acceleration programs will be different in CC'.

What we can say is that *from the point of view of* P Dick, S and Jane may be seen as being at rest with respect to each other in an instantaneous inertial system which can be identified with S. However, if the system S is to be inertial itself, its velocity will be different from that of Dick and Jane immediately before and after the moment it comoves with them. But from the point of view of Dick or Jane, for S to be an instantaneously comoving inertial system, it should move at the same speed either at the time given by the x axis of B or at that given by the x axis of A. That means that if S is an inertial system comoving with B at the time of event $e1$ in Dick's frame, then $e2$ and $e3$ will be simultaneous with $e1$. Of course, this means that only B and S are instantaneously at rest relative to Dick, while A is moving at some non-zero velocity away from both of them. On the other hand, if we require S and A to be instantaneously comoving according to Jane, the events that are simultaneous are $f1$, $f2$, and $f3$, and obviously system B is not at rest with respect to either S or A.

This destroys much of Quattrini's argument. It seems to me that the reason for the introduction of S was to have a means of stopping B and A simultaneously, by sending a signal from S. This would work for motion at constant velocity albeit not the way imagined by Quattrini, because the two systems would receive their signal at the same time in a moving inertial system, not in P. But then the formula for the time difference used by Quattrini is incorrect, as it refers to *equal times in* P. In reality, the situation is worse: there is no inertial system comoving with either A or B at all, in which a signal sent from S would arrive at the same time for both observers or for P. This can be seen from the light rays sent from an S that is at rest w.r.t. to Dick and Jane at $t = 0$, where the left-running ray arrives at t_{l0} at Dick's position, and the right-running ray at t_{r0} at Jane's position, which is definitely *later* in P, but also in B as the picture suggests – the line connecting the arrival events of the two rays is not parallel to the x axis of B near the event labeled by t_{l0}.[10] The situation does not change significantly, if S sends its signals at t_1 in P. Then the left-running signal will arrive at A at t_{Sl0} and the right-running one even leaves the figure before arriving at B.

However, we do not have to send signals to make A and B switch to constant-velocity motion at a fixed time in P. Rather, this is pretty easy. Put an observer, staying at rest in P, at any point in space; each of these is to have a watch and these watches are synchronized in P (via the Einstein procedure). We agree on the time t_2 in advance as the moment when the parents want the kids to shut their engines down. Then the two observers who are next to Dick and Jane at the moment their watches indicate the time t_2 signal them to shut down their engines. Since they are at the positions of Dick and Jane, this takes no time, and we have a procedure that stops their acceleration correctly to make them coast as in Fig. 2. Can this ever lead to a problem with the stopping time in B being smaller than zero? No. Regardless of how fast the acceleration is, we see from the picture that if $t_2 > 0$, then both A and B will have times greater than zero as well.

What if we try to do the stopping instantaneously in S? We could do it the same way as in P, but now our observers have to be at rest in S, i.e., they must move at a fixed velocity v (in P) all the time. If the predetermined time of the two observers next to Dick and Jane was chosen to correspond to event $e1$, then the acceleration will be stopped at events $e1$ and $e3$. In that case, Dick will be at rest in S, but Jane will move to the right. The time formula used by Quattrini does not apply. Moreover, Jane will from that time on age more slowly than Dick, due to the standard relativistic time dilation. If the predetermined time was chosen much smaller, corresponding to event $f1$, then Dick will never start his engine. Jane will stop to be at rest in S but Dick will move to the left in S and therefore age more slowly than Jane as

[10]But it is almost parallel, so a more precise statement requires a calculation.

long as he does.[11]

None of these scenarios corresponds to what Quattrini had in mind. But what he describes does not realize what he had in mind. So no contradiction with Lorentz invariance has been demonstrated.

A few calculations

In fact, the above considerations are completely sufficient to disprove Quattrini's claims, even though they are based on a qualitative approach, the drawing of diagrams. But these diagrams represent the spatiotemporal situation correctly, and we can even base detailed calculations on their geometric features.

I will use this now to derive the correct formulas for the case of a signal being sent from S and for the case of instantaneous termination of acceleration in S when either A or B is at rest there. Since this is a bonus, not necessary for dealing with Quattrini's arguments, readers who are satisfied by the qualitative discussion may skip the calculations and immediately advance to the summary.

To do these calculations, we need to specify the acceleration program (which was not necessary for the drawing of qualitative figures). For simplicity, I will assume the acceleration to be constant during the working time of the engines – in frame P. So the velocity will increase linearly with time. Of course, this means that this kind of acceleration cannot be upheld forever – the speed of light is still a limit.[12] Let Dick start from $x = 0$ at $t = 0$, then his trajectory is described by $x = \frac{g}{2}t^2$ for $t > 0$ (before he switches his engine off), where g is the acceleration in P. Jane is initially located at $x = H$ and moves according to $x = H + \frac{g}{2}t^2$ for $t > 0$. At time t_1, we assume system S to have its origin at $x = \frac{H}{2} + \frac{g}{2}t_1^2$ and a light signal to be sent to the left and to the right from there. We do not even have to specify the speed of S for the following calculations, because light always moves at speed c in P. The two light rays move according to:

$$x_l = \frac{H}{2} + \frac{g}{2}t_1^2 - c(t - t_1)\,,$$
$$x_r = \frac{H}{2} + \frac{g}{2}t_1^2 + c(t - t_1)\,, \tag{3}$$

and to determine their arrival times at the positions of Dick and Jane respectively, we have to intersect their world lines with these straight lines. This gives two quadratic equations (due to our simple choice of acceleration program). Equating x_l with Dick's position, we find

$$\frac{g}{2}t^2 + ct - \frac{g}{2}t_1^2 - \frac{H}{2} - ct_1 = 0\,, \tag{4}$$

solved by

$$t = -\frac{c}{g} \pm \sqrt{\left(\frac{c}{g} + t_1\right)^2 + \frac{H}{g}}\,. \tag{5}$$

[11] Since the observers in S did not have to communicate with either Dick or Jane to implement their procedure, there is also no causality violation. Whether Dick has already powered his engine or or not, does not mean anything to them in terms of their task to send a signal to a spaceship passing by as soon as their clock displays time t_1''.

[12] Constant acceleration in P means increasing proper acceleration, so the procedure would not be healthy for the twins, if kept up for too long.

One of the solutions is positive, the other negative. A look at Fig. 4 tells us that the postive solution is the correct one, because the time at which the left piece of the forward light cone of S hits Dick's world line must be larger than t_1. The negative solution corresponds to a light signal emitted from the continuation of Dick's parabolic world line to negative times and running left to reach S at time t_1. This is not the solution we are looking for. So we take the plus sign to obtain t_{Sl0}.

Equating x_r with Jane's position, we get instead

$$\frac{g}{2}t^2 - ct - \frac{g}{2}t_1^2 + \frac{H}{2} + ct_1 = 0\,, \tag{6}$$

solved by

$$t = \frac{c}{g} \pm \sqrt{\left(\frac{c}{g} - t_1\right)^2 - \frac{H}{g}}\,. \tag{7}$$

Both solutions are positive,[13] but a look at Fig. 4 tells us that here the solution with the minus sign is the correct one, giving t_{Sr0}. We need the solution corresponding to the first intersection with Jane's world line, which is the smaller one of the two. The second solution is unphysical, because there would be only one intersection with a world line curved to the right that nowhere has a slope smaller than one. But in assuming the acceleration to be constant, we obtain a world line that indeed has a slope smaller than one at sufficiently large x. This corresponds to superluminal speeds and the second intersection of our right piece of the light cone of S with Jane's world line would obviously have to be at a position, where this is the case. Our constant-g model of a world line describing acceleration cannot be continued legitimately to these x values. A more realistic model would not have this problem, but might render the calculation of the intersections more difficult.[14]

We then find that the times t_{Sl0} and t_{Sr0} are not equal. Their difference is

$$\begin{aligned} \mathit{\Delta} t = t_{Sr0} - t_{Sl0} &= 2\frac{c}{g} - \sqrt{\left(\frac{c}{g} - t_1\right)^2 - \frac{H}{g}} - \sqrt{\left(\frac{c}{g} + t_1\right)^2 + \frac{H}{g}} \\ &\approx \frac{v_1 H}{c^2}\frac{1}{1 - v_1^2/c^2} + \frac{H^2 g}{4c^3}\frac{1 + 3v_1^2/c^2}{\left(1 - v_1^2/c^2\right)^3}\,, \end{aligned} \tag{8}$$

where the first line is exact and the second is an expansion valid for $H \ll c^2/g$ and taken to quadratic order in H. $v_1 = gt_1$ is the speed of system S on emission of the light signals, i.e., the speeds of B and A are larger, when Dick and Jane switch off their engines. Moreover, they are different from each other, as $\mathit{\Delta} t \neq 0$, so the switching-off does not happen at the same time in P.

It is interesting to compare $\mathit{\Delta} t$ with the result from Eq. (2). If we drop the term quadratic in H, we have $\mathit{\Delta} t = \gamma(v_1)^2 v_1 H/c^2$ which looks similar to that result, except that we have velocity v_1 instead of v and a factor of γ^2 instead of γ. It would be typical for a dabbler in special relativity to interpret the result Eq. (8) as an approximation to the exact result from Eq. (2), arguing that for short times the stopping procedure via a light signal is essentially equivalent to stopping at a fixed time and that the time difference in primed coordinates must,

[13] Unless H/g is too large, in which case there is no solution – the light signal never catches up with the accelerating observer.

[14] We can always solve a quadratic equation explicitly, but may not be able to find analytic solutions for more involved world line equations.

due to time dilation, just be $1/\gamma$ times the result in times of P, so the two results actually agree up to the difference in velocities (v versus v_1) which should be also small... But this argument has no basis. The time dilation formula producing a factor of $1/\gamma$ between the clock rates of times in P and times in C holds only, if the time interval is measured at a fixed position in C.[15] But Δt is a time interval between two events at different positions in all inertial systems considered. Normally, they are even spacelike, so they cannot be at the same position in *any* inertial system. Therefore, there is no room for application of the standard time dilation formula here.[16] Moreover, Δt is the difference between the switch-off times of the engines in system P. This time was assumed to be zero to derive Eq. (2)! One simply should not let oneself being seduced by a superficial similarity of formulas into believing in physical equivalence.

Finally, let us consider the case where Dick's and Jane's stopping is prompted by two observers at rest in S at a predetermined time. There are two interesting cases. The stopping time can be chosen as the time of event $e1$ in Fig. 4, then Dick will have the velocity of S on shutting down his engine, or else stopping is at the time of event $f1$, then Jane will coast at the velocity of S after shutting down her engine. Let us calculate the times $t(e_3)$ and $t(f3)$, at which Jane will switch off her engine in the first and Dick will switch off his engine in the second scenario. This is again easy with a constant acceleration g.

In the first scenario, Dick shuts down his engine at event $e1$ at time t_1 and position $x(e_1) = \frac{g}{2}t_1^2$. His velocity is $v = gt_1$. Jane is stopped at event $e3$ which is simultaneous with $e1$ in S. Because the slope of lines of simultaneity of S is v/c, the time in P is given by

$$ct(e3) = ct_1 + \frac{v}{c}\left(x(e3) - x(e1)\right) = ct_1 + \frac{v}{c}\left(H + \frac{g}{2}t(e3)^2 - \frac{g}{2}t_1^2\right) . \tag{9}$$

This again is a quadratic equation, and the relevant solution for $t(e3)$ is

$$t(e3) = \frac{c^2}{vg} - \sqrt{\left(\frac{c^2}{vg} - t_1\right)^2 - \frac{2H}{g}} . \tag{10}$$

When the square root can be expanded, which is the case for sufficiently small H/g, this turns into

$$\begin{aligned} t(e3) - t_1 &\approx \frac{H}{g(c^2/(vg) - t_1)} + \frac{H^2}{2g^2\left(c^2/(vg) - t_1\right)^3} \\ &= \frac{vH}{c^2}\frac{1}{1 - v^2/c^2} + \frac{H^2 g}{2c^3}\frac{v^3}{c^3}\frac{1}{\left(1 - v^2/c^2\right)^3} \end{aligned} \tag{11}$$

Comparing this with Eq. (8), we see that the results are equal to lowest order in v/c but not at the next, the cubic order. Also, they agree to linear order in H but not at quadratic order. This shows that for sufficiently small velocities v or distances H, sending a light signal from a comoving inertial system intermediate between the twins' positions will approximately realize simultaneous stopping in that frame, but not, of course simultaneous stopping in P, which would be necessary to use Bouhgn's results.

Now have the stopping time chosen so that it is Jane that will come to rest in S. Here we have to distinguish two cases. The event $f3$ at which Dick will be stopped is at $x(f3) = 0$, if

[15] In deriving that formula, one has to assume the clock to be at rest in one of the systems.

[16] Instead, the full Lorentz transformations must be used.

$t(f3) < 0$ and it is at $x(f3) = \frac{g}{2}t(f3)^2$, if $t(f3) > 0$. What are the precise conditions for the first case to happen? We have (see Fig. 4)

$$\begin{aligned} ct(f3) &= ct_1 + \frac{v}{c}\left(x(f3) - x(f1)\right) = ct_1 + \frac{v}{c}\left(x(f3) - \frac{g}{2}t_1^2 - H\right) \\ &= ct_1 - \frac{v}{c}\left(\frac{g}{2}t_1^2 + H\right) + \frac{v}{c}x(f3)\,, \end{aligned} \tag{12}$$

and the last term of the second line is alway greater than or equal to zero. So a necessary condition for $t(f3) < 0$ is

$$ct_1 - \frac{vg}{2c}t_1^2 - \frac{vH}{c} < 0\,. \tag{13}$$

Using $v = gt_1$, dividing by $t_1 > 0$ and multiplying by c, we get

$$c^2 - gH < \frac{g^2}{2}t_1^2\,. \tag{14}$$

Since our assumption of constant acceleration restricts t_1 to times smaller than c/g, the right hand side is necessarily smaller than $c^2/2$. Therefore, if $gH < c^2/2$, condition (14) is never satified. We need either sufficiently large distance H between the twins or sufficiently large acceleration g to obtain the effect that simultaneously stopping Jane and Dick in S will lead to Dick being stopped in P even before having started. Note that if $gH > c$, then the condition is always satisfied and it is *inevitable* that if Jane is stopped sometime inside the legitimate t_1 interval, Dick has not yet fired up his engine in the frame S.

Suppose now that the condition is not fulfilled, so Dick has already started according to S, when Jane is stopped. Then the calculation is fully analogous to that of the case where Dick is stopped at event $e1$. We have

$$ct(f3) = ct_1 + \frac{v}{c}\left(x(f3) - x(f1)\right) = ct_1 + \frac{v}{c}\left(\frac{g}{2}t(f3)^2 - \frac{g}{2}t_1^2 - H\right)\,. \tag{15}$$

This is solved by

$$t(f3) = \frac{c^2}{vg} - \sqrt{\left(\frac{c^2}{vg} - t_1\right)^2 + \frac{2H}{g}}\,. \tag{16}$$

When the square root can be expanded, which is the case for sufficiently small H/g, this turns into

$$\begin{aligned} t(f3) - t_1 &\approx -\frac{H}{g(c^2/(vg) - t_1)} + \frac{H^2}{2g^2\left(c^2/(vg) - t_1\right)^3} \\ &= -\frac{vH}{c^2}\frac{1}{1 - v^2/c^2} + \frac{H^2 g}{2c^3}\frac{v^3}{c^3}\frac{1}{\left(1 - v^2/c^2\right)^3}\,, \end{aligned} \tag{17}$$

where we have a minus sign in front of the first term, because $t(f3)$ must be smaller than t_1. Here a similar interpretation as for Eq. (11) is possible, because H (or v) can be arbitrarily small.

The more interesting case is of course the one, where $t(f3)$ does indeed become negative. Then the second form of Eq. (15) does not hold, but the first form is of course still valid with $x(f3) = 0$, which leads to an explicit equation for $t(f3)$:

$$ct(f3) = ct_1 + \frac{v}{c}\left(-\frac{g}{2}t_1^2 - H\right)\,, \tag{18}$$

hence

$$t(f3) - t_1 = -\frac{v}{c^2}\left(\frac{g}{2}t_1^2 + H\right) = -\frac{v}{c^2}\left(\frac{v^2}{2g} + H\right) = -\frac{v}{gc^2}\left(\frac{v^2}{2} + gH\right)$$
$$< -\frac{v}{g} = -t_1\ , \tag{19}$$

where we have used that condition (14) implies $v^2/2 + gH > c^2$. Note that this scenario is not achievable via signal sending from S to any useful approximation. While we have known that already from the discussion of Fig. 4, demonstrating that any signal sent from S after $t = 0$ cannot arrive at Dick's position at a time $t < 0$, we can read it off here from the side conditions. v is always smaller than c, so H must be sufficiently large for condition (14) to be satisfied. But then the expansions used to show approximate equivalence of the signal sending and the heralded stopping scenarios do not work anymore. In these expansions, we must have H sufficiently small, which will invalidate condition (14). So Quattrini's argument fails on that account, too.

Summary

It has been shown that special relativity describes all aspects of the situation of the equally accelerating twins consistently. While some of these may be surprising at first sight, they all find a natural explanation in the kinematical relationships following from the structure of Minkowski spacetime. There is no "breaking of Lorentz invariance" ar anything of similar dramatic appeal.

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