Why is there a collapse of the wave function?

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The opinion has been expressed in some of the preceding RG discussions that the collapse of the wave function is not part of quantum mechanics, which would rather be comprised of the equations of motion of the wave function, i.e., the Schrödinger equation. Introducing the collapse would, in that opinion, be only an interpretative addendum that may also be wrong, and proofs could be sought via thought or real experiments demonstrating the fallacy of the concept. And indeed, there seems to be an interpretation of quantum mechanics, the many-worlds or relative-state interpretation, which gets by without a collapse.

Derivation of the collapse

I would like to demonstrate that, on the contrary, any utilization of a theory that makes the same predictions as quantum theory, inevitably leads to a state collapse of some sort, independent of the interpretation. After having given my arguments and discussed some examples, I will explain how this is also true in two non-standard interpretations of quantum mechanics including the many-worlds interpretation.

My demonstration will be based on some of the basic axioms of quantum mechanics, i.e., I will *derive* the collapse from these axioms, and I think this is the way it entered the theory originally. I will not attempt to give a complete axiomatization of quantum mechanics, rather restrict myself to the small number of axioms that I actually need.

These axioms are:

1. A (pure) state of a quantum system is describable by a Hilbert space vector.[1]
2. Observables are described by self-adjoint operators[2] on this Hilbert space and ideal measurements of an observable yield eigenvalues of its associated operator. The probability or probability density to obtain a particular eigenvalue a in measuring the observable to which the operator A is associated, is given by $w(a) = |\langle a|\psi\rangle|^2$, if $|\psi\rangle$ is the state of the system just before the measurement and $|a\rangle$ is the eigenvector corresponding to the eigenvalue, both with norm one.[3] This holds, if the eigenvalue is nondegenerate. Else, we have $w(a) = \sum_{n=1}^{f} |\langle a,n|\psi\rangle|^2$, with $\{|a,n\rangle, n = 1\dots f\}$ denoting an orthornormal basis of the eigenspace of A corresponding to the eigenvalue a.
3. Results of *performed* measurements correspond to real properties of the measured system immediately after the measurement.

The first two axioms do not need any further comments, they are part of the basic mathematical setup of the theory and interpretation independent. I employ the framework of the Schrödinger picture. In other formulations of quantum mechanics, modified but equivalent versions of these axioms may apply.

[1] This is an essential element of the starting definition of the concept of *state*. Mixed states are introduced only later in quantum mechanics, so the qualifier "pure", anticipating an extension of the definition of state, is put into parentheses here.

[2] I will not enter into the ramifications of rigged Hilbert spaces here.

[3] $w(a)$ is a probability, if the eigenvalue belongs to the discrete part of the spectrum of A and a probability density if it belongs to the continuous part of the spectrum of A (either of which can be empty, but not both).

The third axiom is rarely mentioned. But it is relevant in view of the peculiarities of quantum mechanics leading to questions about what is real and what is not. There are interpretations of quantum mechanics denying the simultaneous reality of positions and momenta (more precisely, their components in a given direction). But quantum theory is a theory meant to describe the real world. So we should state what is to be considered real. Note that in classical physics, we do not have this kind of doubt. We assume that measurements can be made without significantly perturbing a system and that therefore the result of a measurement reveals a real property that the system had at the instant of the measurement. There is no need to specify whether this is just before or immediately after the measurement, because we do not have to care whether the measuring apparatus has been separated from the system or not. The measured property is present with and without the apparatus.

However, even in classical physics we cannot state the measured property to be a system property *indefinitely after* the measurement. Having measured the position of a moving particle, we may say it had that position at the time of the measurement or during an infinitesimal time interval about that time, but of course it will be elsewhere some time later, as it is moving, and if we do not measure its velocity in addition, we cannot say precisely where.

In quantum physics, the process of measurement often perturbs the system significantly. So we cannot claim that the measured property is one that applies to the system *before* we coupled the measuring apparatus to it. But if the measuring apparatus is to serve its purpose, we must believe that the read-off value of the measured property is a real property of the system at *some* time. We would like to be this *at* the instant of the measurement. However, at that instant the measuring apparatus is still coupled to the system (and their common state is a superposition). To obtain a definite value of the measured quantity, we must abstract from the apparatus and consider the system separated from it. A good mental picture for this is to say we have the measured result immediately *after* the measurement (having accounted for the – ideally infinitesimal – time it takes to separate the apparatus from the system[4]).

In order to render this somewhat abstract discussion more concrete, let us consider how this works for some examples. First assume that we measure the z component of the spin of an electron using a Stern-Gerlach apparatus. Then the statement is that immediately after having left the apparatus, the electron will *have* the measured component (spin up, if it comes out in the upper track, spin down if it comes out in the lower track). In fact, if we make sure that no external magnetic field perpendicular to the measured spin direction is present, then "immediately after" can be pretty long. For quantities that commute with the Hamiltonian, the measured property will prevail as long as we desire. And this can be easily checked by sending the electron through another Stern-Gerlach magnet oriented the same way as the first – it will always come out through the same slot as it did with the first.[5] In the second example, assume we measure the position of a photon or neutron by having it hit a screen after having gone through some experimental arrangement (a two-slit experiment, say). Then the statement of

[4] Note that this may be seen as a reason for time asymmetry of measurements. We grant that the measuring apparatus may perturb the system significantly on being coupled to it, but we are willing to accept that the decoupling of the apparatus from the system may be done without perturbation, in theory. This is in accord with the projection postulate that I am going to justify. Since the system is in an eigenstate of the measured observable after measurement, removal of the apparatus need not perturb it.

[5] The practical implementation of this is a bit more complicated than this simple discussion suggests, as we do not see the electron and observing it usually means its absorption. But we can repeat the experiment with varying numbers of Stern-Gerlach magnets arranged in a series, and will always find the electron to hit the final screen at a point where it can have gotten only by taking the same track in each of the magnets. We may also do the experiment with more easily observable particles such as spin $\frac{1}{2}$ atoms.

axiom 3 is difficult to assert directly, because measurement means destruction of the state by interaction with the potographic plate or the scintillator material, or whatever. Nevertheless, the mental splitting of the position measurement into two stages, the first being a state change into a state of definite position and the second being a series of (chemical) interactions which destroy that position eigenstate, will not lead to incorrect statements about measured properties. Therefore, axiom 3 seems acceptable in that situation, too. In fact, it is often implicitly used in discussions of measurements.

We may ask in this context, to which observer "immediately after" refers, given that in special relativity this notion does not make sense, when referring to two distant events. Of course, this is a problem only for nonlocal measurements, but as we shall see, those are a possibility in quantum mechanics. The answer is that a good starting point for a frame of reference is a frame of the observer making the measurement. It will turn out that not everything may be formulated in a Lorentz invariant manner but that no contradictions can be derived from using different frames.

Now let us put the three axioms to work. Assume we carry out a measurement that can be done repeatedly on the particle, i.e., a measurement that does not lead to absorption without "reusability".[6] We measure the same observable on the particle twice, three times or more often, in quick temporal succession. For simplicity of discussion and language, I will restrict myself (at this point) to the discrete nondegenerate case. Let the initial state be $|\psi\rangle$. The first measurement of A gives the eigenvalue a with probability $|\langle a|\psi\rangle|^2$. What does the second measurement give? Axiom 3 tells us that the particle immediately after the first measurement really has the property a, so if the second measurement follows fast enough, we must get a again, with probability one. The third measurement will give a again, and so on. Now what can the state $|\psi_1\rangle$ of the system after the first measurement be? Since on measuring A again we get the value a with probability one, we must have $|\langle a|\psi_1\rangle|^2 = 1$, and since both $|a\rangle$ and $|\psi_1\rangle$ are normalized, this means we must have $|\psi_1\rangle = \mathrm{e}^{\mathrm{i}\alpha}|a\rangle$, i.e., $|\psi_1\rangle$ is an eigenstate of the operator A corresponding to the eigenvalue a. Obviously, we have that the states $|\psi_2\rangle$, $|\psi_3\rangle$, etc. are all equal to $|a\rangle$ up to a phase factor.[7]

Actually, this shows that axiom 3 is nothing but a sanity requirement for quantum theory, if the theory is to describe the real world: if a measured property is real, remeasuring it without leaving the system time to evolve to a different state must give the same property. Any rational theory must require this.

But what does it mean? The initial state $|\psi\rangle$ need not have anything to do with the eigenstates of the observable A, whereas the first state $|\psi_1\rangle$ obtained after measurement of A must be an eigenstate. Well, *this is the collapse.*

Why such a dramatic name? Clearly, we can measure many quantities in a decently short time. The change from $|\psi\rangle$ to $|\psi_1\rangle$ must happen in that short time interval, no matter what. So it can be a dramatic time evolution. For example, if we measure photons coming from a quasar that appears as a double image by gravitational lensing and where there is sufficient coherence between the two images so an interference pattern can arise, then the wave function at some time may have contained wave packets that were light years apart. Now, at the time of measurement,

[6] An example would be the Stern-Gerlach experiment, embellished by some means to detect whether the particle (an atom or ion) is in the upper or lower track, without absorbing it.

[7] If A does not commute with the Hamiltonian, this gives rise to the so-called *quantum Zeno effect*, i.e., the particle is kept in the eigenstate $|a\rangle$ by repeated measurement, even though under ordinary time evolution, it would evolve away from it.

its wave crests may still have lateral extensions at the kilometer size level, as the quasar is visible everywhere on a hemisphere of the Earth. So the interfering wave may be huge. But we know that only photons in the same single-particle states interfere and these will arrive sequentially as small blobs on the photographic plate. Hence, measurement may reduce a kilometer-sized (or larger) wave function to a micrometer-sized blob in a matter of microseconds. Obviously, the speed of light is not a limit for the speed of the collapse. This makes it difficult to develop a dynamic description of the change of state during the collapse, because faster-than-light changes tend to clash with special relativity.

Which also provides an argument against the ontic nature of the wave function. If it were ontic, then special relativity would have to be wrong at the ontic level. It might still survive as a phenomenological or "epistemic" description.

Nevertheless, the desire to have a more detailed dynamic account of the collapse has led to some attempts at such an approach, such as the Ghirardi-Rimini-Weber theory [1]. Yet, nobody has succeeded in developing a detailed dynamical description that reproduces the predictions of quantum mechanics exactly. The predictions of the GRW theory deviate from those of quantum mechanics at long times and large system sizes, but the parameters of the theory have been chosen cunningly so that it is impossible at present to distinguish empirically between standard quantum mechanics and the GRW theory. My guess is that if a distinction will ever be possible, the predictions of quantum mechanics will prevail and those of the GRW theory fail. But that is just an opinion.

There are also good reasons to believe that a dynamical description of the collapse will never be possible. In a way, it is very easy to understand where it comes from: it is a consequence of the interpretation of scalar products involving the state vector $|\psi\rangle$ (e.g., $\langle x|\psi\rangle = \psi(x)$ or $\langle a|\psi\rangle$) as probability amplitudes.

In classical physics, we sometimes deal with master equations describing the time evolution of probability densities. Whenever we learn something new about the system, e.g., by a measurement (that in classical physics may be assumed to not perturb the system at all) the describing probability density changes, but this change does not follow from the master equation. There are no equations for it. Probabilities depend on knowledge, and we are accustomed to the fact that when we remove the covering cup from a thrown die the probability for the shown number immediately changes from 1/6 to 1 (and for the five other numbers to 0), without any real change to the die. We do not feel the need for a dynamical description of that collapse of probability, because the probability is not the fundamental dynamical variable. For classical systems, the fundamental variables are positions and momenta, and those do not change on knowledge-induced changes of the probability.

In quantum mechanics, however, the wave function has the double role as a probability amplitude and a description of the quantum state. The evolution of the latter is described by the Schrödinger equation, the analogue of the classical master equation, and as in classical mechanics we do not have dynamical equations for the change of probabilities on information gain by a measurement. We just have rules for getting from the probabilistic description before to the probabilistic description after, the measurement. One interpretation of quantum mechanics, the ensemble interpretation [2], denies the applicability of wave functions to single quantum systems, reducing them to descriptions of ensembles of systems. Then the collapse is a natural phenomenon, just as in classical physics, and of course not a dynamical process. What may seem less plausible is that followers of the ensemble interpretation have not made enormous efforts so far to find a quantum theoretical description that *does* apply to single systems. It would be natural to expect such attempts in view of the fact that in the ensemble interpretation such a description is missing.

To summarize, what can be said is that the system state turns, between starting and completing a measurement, from an initial state $|\psi\rangle$ to a final state $|\psi_1\rangle$, often completely different from the initial state, and the collapse abstracts from the details of this transition. However, this change of state[8] *must* happen on measurement in quantum theory, regardless of interpretation, if the three axioms hold.

The only way to avoid it would be to drop one of the axioms. Since the first two axioms are essential ingredients of the mathematical structure of the theory, they cannot easily be abandoned. So we would have to give up axiom 3 in order to have no collapse. How could we do so? If we assume that the result of a measurement does not correspond to a real property of the system immediately after measuring, how are we to explain the experimental fact that the same result is obtained on immediate repetition of the measurement?[9] Persistence of results and objectivity are usually associated with reality. Assuming that axiom 3 does not hold would mean denying reality to observations that can be consistently verified by repetition.

The projection postulate and applications

Nevertheless, axiom 3 is rarely spelled out. Rather, it is replaced, in a formal axiomatization of quantum mechanics, such as von Neumann's, by the projection postulate. This has the economic advantage that it is not necessary to *derive* on each occasion that the state of a system immediately after measurement is an eigenstate of the measured observable[10] (which is what I did above), but it is just required as an axiom, roughly equivalent to the axiom 3 given above. The projection postulate, which we might call axiom 3' (replacing axiom 3) states that, if we measure an observable A, the resulting state vector immediately after the measurement (in the sense discussed above) is obtained by projecting the state vector from before the measurement on the eigenspace of A corresponding to the measured eigenvalue a, and that the factor needed to normalize the projection to one is the probability amplitude for the measurement of a.[11] Normalization means conditioning the probability on the measurement of a and is necessary, if the state vector after measurement is required for further predictions.

Formally, the measurement process is thus described by the sequence:

$$|\psi\rangle \to \sum_{n=1}^{f} |a,n\rangle \langle a,n|\psi\rangle \to |\psi_1\rangle = \frac{1}{\left(\sum_{n=1}^{f} |\langle a,n|\psi\rangle|^2\right)^{1/2}} \sum_{n=1}^{f} |a,n\rangle \langle a,n|\psi\rangle \ . \tag{1}$$

Note that in the nondegenerate case, the system completely "forgets about" its intial state through the measurement, because we simply have $|\psi_1\rangle = \mathrm{e}^{\mathrm{i}\alpha} |a\rangle$. In the degenerate case, i.e., when the sum contains more than one term, the state vector after measurement carries some memory of the initial state, because the direction of this vector in the subspace depends on the state $|\psi\rangle$. In the nondegenerate case, the only information about the initial state remaining in the final state is the phase factor and that is physically irrelevant.[12]

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[8] Which sometimes isn't a change: if by accident $|\psi\rangle$ is already an eigenstate of A, then the collapse does nothing.

[9] Besides being a sanity requirement, this also has been verified experimentally.

[10] I.e., its associated operator. This is a slight abuse of language.

[11] This has to be refined for the measurement of single-particle properties with many-particle states. See below.

[12] Of course, the "length of the projection" is also a relevant information, giving the probability amplitude, but that information is not present anymore in the normalized final state.

So the projection postulate provides a concise description of measurements, disregarding the details of the interaction between apparatus and quantum system. Whenever a more detailed description is desired, the combination of measuring apparatus and system to be measured may be considered and described by a Schrödinger equation in a larger Hilbert space. This will result in a unitary evolution of the combined system, comprising all possible outcomes of the measurement and their probability amplitudes, but no decision as to which result the measurement actually had.

Let us now apply the projection postulate to a few standard cases to see how it works.

Double slit experiment: Assume $|\varphi_1\rangle$ and $|\varphi_2\rangle$ to be the two partial state vectors corresponding to a quantum particle going through slits 1 and 2, respectively, of a Young-type double slit setup. The total state then is the superposition

$$|\varphi\rangle = \frac{1}{\sqrt{2}}\left(|\varphi_1\rangle + |\varphi_2\rangle\right) . \tag{2}$$

We measure the position of the particle at the screen, so we have to project on an eigenstate of the position operator, which is achieved using the projection operator $|x\rangle\langle x|$, corresponding to the measured position value x:

$$|\varphi\rangle \to |x\rangle\langle x|\varphi\rangle = \frac{1}{\sqrt{2}}|x\rangle\left(\langle x|\varphi_1\rangle + \langle x|\varphi_2\rangle\right) = \frac{1}{\sqrt{2}}\left(\varphi_1(x) + \varphi_2(x)\right)|x\rangle \to |x\rangle , \tag{3}$$

which tells us that after measurement of the particle at position x (the eigenvalue), the state will be $|x\rangle$ and this will occur with probability density $w(x) = \frac{1}{2}\left|\varphi_1(x) + \varphi_2(x)\right|^2$, which is proportional to the intensity of the observed interference pattern.

Double slit and which-way information: It is possible to do the double slit experiment with atoms that store which-way information in an internal state (excited via a microwave-field, the quanta of which have much lower energy than (low quantum number) atomic electronic transitions) [3]. Let us write the internal state, which has two possible values, in spin notation. Then the two partial states are $|\varphi_1\rangle|\uparrow\rangle$ and $|\varphi_2\rangle|\downarrow\rangle$, the total state being the superposition

$$|\tilde{\varphi}\rangle = \frac{1}{\sqrt{2}}\left(|\varphi_1\rangle|\uparrow\rangle + |\varphi_2\rangle|\downarrow\rangle\right) . \tag{4}$$

A position measurement produces the projection

$$|x\rangle\langle x|\tilde{\varphi}\rangle = \frac{1}{\sqrt{2}}|x\rangle\left(\langle x|\varphi_1\rangle|\uparrow\rangle + \langle x|\varphi_2\rangle|\downarrow\rangle\right) = \frac{1}{\sqrt{2}}\left(\varphi_1(x)|\uparrow\rangle + \varphi_2(x)|\downarrow\rangle\right)|x\rangle . \tag{5}$$

Since now the position eigenstate is not multiplied by a number but by a linear combination of the two internal states, the calculation of the probability amplitude is slightly different. There are various ways to proceed, but it is simplest to apply a rule that works in all cases. That rule says that to obtain the probability (density) of a single-state measurement given a multiple-state Hilbert vector, we should take the density operator describing the projected state and "trace out" all non-measured states. The probability (density) is thus obtained from a sum over non-measured states.[13] The relevant density operator here is:

$$\varrho = \frac{1}{2}\left(\varphi_1(x)|\uparrow\rangle + \varphi_2(x)|\downarrow\rangle\right)\left(\varphi_1^*(x)\langle\uparrow| + \varphi_2^*(x)\langle\downarrow|\right)|x\rangle\langle x| \tag{6}$$

[13] This rule results from a rather straightforward generalization of the method to calculate marginal probability distributions from joint probability distributions.

and the sought-for trace is given by

$$\mathrm{tr}_{\,\mathrm{int}}(\varrho) \equiv \langle\uparrow|\varrho|\uparrow\rangle + \langle\downarrow|\varrho|\downarrow\rangle = \frac{1}{2}\left(|\varphi_1(x)|^2 + |\varphi_2(x)|^2\right)|x\rangle\langle x| \ . \tag{7}$$

The prefactor of $|x\rangle\langle x|$ is the probability density of measuring the particle at x and, as expected, this does not form an interference pattern, because the presence of which-way information excludes interference.

Double slit and detector at one slit: Instead of storing the which-way information in an internal degree of freedom of the particles, we may simply put a detector behind one slit that fires whenever a particle comes through this slit. Let us assume the detector D_1 to be located at slit 1 and that it is a perfect detector, i.e., that it fires in *all* cases when a particle gets into it. The projections produced by measurements of this detector are then described by the operators $|\varphi_1\rangle\langle\varphi_1|$ and $1 - |\varphi_1\rangle\langle\varphi_1|$, respectively, the first corresponding to the detector firing on passage of a particle, the second to it not firing (because the particle passed through slit 2). If the detector fires, the state vector (given in Eq. (2)) of the system collapses as follows:

$$|\varphi\rangle \to |\varphi_1\rangle\langle\varphi_1|\varphi\rangle = \frac{1}{\sqrt{2}}\left(1 + \langle\varphi_1|\varphi_2\rangle\right)|\varphi_1\rangle \to |\varphi_1\rangle \ . \tag{8}$$

Note that the state after firing is always $|\varphi_1\rangle$, even if $|\varphi_2\rangle$ is not orthogonal to $|\varphi_1\rangle$. This would mean that the detector fires sometimes due to a particle having come through slit 2. The ideal case is, however, the one where the two partial waves have no overlap at the time they pass the slits, so they are orthogonal.[14] And it is only in this case that $1/\sqrt{2}$ is the correct normalization factor for the superposition of the two partial states.

If the detector does not fire and now assuming orthogonality of the two states, we find

$$|\varphi\rangle \to \left(1 - |\varphi_1\rangle\langle\varphi_1|\right)|\varphi\rangle = \frac{1}{\sqrt{2}}|\varphi_2\rangle \to |\varphi_2\rangle \ , \tag{9}$$

so only the branch $|\varphi_2\rangle$ survives.[15] Note that in the case of a non-firing detector, we have no precise knowledge about *when* the state has been measured and, thus, collapsed. If, moreover, the detector is not perfect, i.e., does not capture 100% of the particles passing through it, we do not even know, *whether* it has collapsed the wave function of the quantum particle when it does not fire.

That the precise point in time of the collapse cannot be determined, need not come as a surprise. Our derivation above only proves the *fact* of a collapse. A collapse must have happened, if the initial state was not an eigenstate of the observable, by the time the result of the measurement is fixed. Since we do not have a dynamical description, we cannot be much more precise.[16]

What is worse, the experiment can be described in an alternative way that postpones the collapse to the time the particle position is measured, so even a firing detector need not be taken as a

[14] They will then remain orthogonal, even though they must develop some overlap to produce an interference pattern (in the absence of D_1). Interference requires only that $\varphi_1^*(x)\varphi_2(x)$ is nonzero somewhere. The integral of this quantity over all of space can still vanish.

[15] Nonorthogonality would here imply that the state after the collapse still contains a $|\varphi_1\rangle$ contribution.

[16] Attempts at giving a precise instant of the collapse may result in counterfactual language ("if the measurement takes a millisecond, the collapse must take place between t_1 and t_2") and that is dangerous in discussions of quantum mechanics.

failsafe signature that the collapse has already happened. This occurs, if we try to give a more detailed account of the interactions involved in the measurement by considering the combined state of particle and detector. Let us call $|D_1\rangle$ the state of a detector that has not fired, and $|D_1^*\rangle$ its "excited state", when it has fired. Then we may describe the evolution as follows:

$$\begin{aligned}
|\varphi\rangle\,|D_1\rangle = \frac{1}{\sqrt{2}}\left(|\varphi_1\rangle + |\varphi_2\rangle\right)|D_1\rangle \underset{\text{unitary evol.}}{\longrightarrow} & \frac{1}{\sqrt{2}}\left(|\varphi_1\rangle\,|D_1^*\rangle + |\varphi_2\rangle\,|D_1\rangle\right) \\
\underset{\text{pos. measurem.}}{\longrightarrow} & \frac{1}{\sqrt{2}}\,|x\rangle\left(\langle x|\,\varphi_1\rangle\,|D_1^*\rangle + \langle x|\,\varphi_2\rangle\,|D_1\rangle\right) \\
= & \frac{1}{\sqrt{2}}\,|x\rangle\left(\varphi_1(x)\,|D_1^*\rangle + \varphi_2(x)\,|D_1\rangle\right)\ , \qquad (10)
\end{aligned}$$

where "unitary evol." refers to evolution according to the Schrödinger equation of the combined particle-detector system. The collapse arises only on the final position measurement, and evaluating the probability density via tracing the detector states out of the final state gives $w(x) = \frac{1}{2}\left(|\varphi_1(x)|^2 + |\varphi_2(x)|^2\right)$, i.e., the result formally looks like that of Eq. (7). We clearly do not have interference. Nevertheless, on evaluation the formulas will behave differently. In Eq. (7), x is a position on the screen and both terms of the formula can contribute to the intensity at the same point. In the experiment discussed here, x is either a position in the detector,[17] in which case $|\varphi_2(x)|$ is zero, or it is a position on the screen, in which case $|\varphi_1(x)|$ vanishes (assuming the detector blocks all the particles it detects). Measuring a particle position inside the detector may be an experimental challenge. If the detector is small enough, we may consider its firing already to be a position measurement, then the final position measurement happens together with the determination of the which-way information for a firing detector and only later for a non-firing one.

If we consider only positions on the screen (i.e. measure positions only when the detector has not fired), then the probability distribution will be the one-slit function $|\varphi_2(x)|^2$.

Finally, I should add that the second description assumes a perfect detector. If the detector has a finite detection probability, the combined particle-detector system cannot be described by a wave function anymore, we have to use a density operator description.

Entanglement: A particularly interesting situation is the case of an entangled two-particle state. A few words on entanglement may be in order. If we wish to describe many-particle states, we need a different Hilbert space than for single-particle states. By a little reflection, it is possible to convince oneself that this Hilbert space can be constructed as a tensorial product from the constituent Hilbert spaces.[18] Product states of the basis states of the constituent Hilbert spaces provide a basis of the total Hilbert space. Moreover, these product states are examples of many-particle Hilbert vectors. But they are not the only examples. Any superposition of product states is a legitimate state of the composite system, and any such superposition that is not a product state itself is an entangled state.

Entanglement of two or more particles is not interaction and it is not a "coupling". It is a result of the fact that the time evolution of a product state under a Hamiltonian that is not just a sum of constituent system Hamiltonians but contains interaction terms will not

[17] Remember, x is a *measured* position.

[18] Hermitean operators have complete sets of orthonormal eigenfunctions. The product of two commuting Hermitean operators acting on the properties of either particle, respectively, is Hermitean, and the products of the eigenvectors of the original operators are eigenvectors of their product.

remain a product state, in general. The Einstein-Podolsky-Rosen paper [4] possibly is the first that discusses entanglement as a property of two particles that "have previously interacted".[19] Whether such an interaction is really necessary, appears dubious. In statistical mechanics, the symmetry properties of the many-particle wave function under particle exchange lead to entangled states without any direct necessity of interaction. In any case, there is a major difference between interaction and entanglement. *Entanglement* of two particles cannot be detected by measurement on one of them *alone*. On the other hand, *interactions* of one particle with another *may* be detectable by measurements on the first particle only.

One of the fun entangled states to consider has the form

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|\varphi_A, \uparrow\rangle\, |\varphi_B, \downarrow\rangle - |\varphi_A, \downarrow\rangle\, |\varphi_B, \uparrow\rangle \right) , \tag{11}$$

where the arrows designate spin $\frac{1}{2}$ components in a chosen direction (the z direction by convention), so this is a singlet state, and φ_A and φ_B characterize the spatial part of the state vector for the particles A and B. An interesting aspect of this state is that it is *isotropic* in spin space, i.e., the specification that the spin labels refer to the z direction would have been unnecessary, because if it denotes another direction, that would not change the state.[20]

Should you contemplate what is the state of each particle in this two-particle wave function – neither of the particles has its own (pure) state! In standard quantum mechanics, the wave function provides the maximum possible information about a system.[21] So given our two-particle state, we have complete information about the two-particle system, but we do not have complete information about either of its constituent particles. What would we get, if we prepared many identical two-particle states (11) and tried to find out the state of the A particle in them by making all kinds of measurements on it, ignoring particle B? We would find that A is in a mixed state, given by the density operator

$$\varrho_A = \frac{1}{2} \left(|\varphi_A, \uparrow\rangle \langle \varphi_A, \uparrow| + |\varphi_A, \downarrow\rangle \langle \varphi_A, \downarrow| \right) , \tag{12}$$

but a mixed state does not contain maximal possible information about a system.[22] It is actually a shortcut for saying the system is with a given probability in the state corresponding to the first diagonal element of the density matrix (if it is diagonal), with another given probability in the state corresponding to the second diagonal element, and so on.

Now what happens if we measure the spin component of particle A along the direction that we specified via the up and down components in the formula? The result of such a measurement can be either "up" or "down", i.e., the measurement collapses the wave function according to one of the following two cases

$$\begin{aligned} |\psi\rangle \to |\uparrow\rangle_A \langle\uparrow|_A \frac{1}{\sqrt{2}} \left(|\varphi_A, \uparrow\rangle\, |\varphi_B, \downarrow\rangle - |\varphi_A, \downarrow\rangle\, |\varphi_B, \uparrow\rangle \right) &= \frac{1}{\sqrt{2}} |\varphi_A, \uparrow\rangle\, |\varphi_B, \downarrow\rangle \\ &\to |\varphi_A, \uparrow\rangle\, |\varphi_B, \downarrow\rangle , \end{aligned} \tag{13a}$$

[19] The methods used to create entangled particles normally involve their interaction at some stage. However, there is "entanglement swapping" which allows to entangle two particles that have never interacted with *each other*.

[20] This can be proved by applying a spin rotation to the state and demonstrating that the resulting state is identical to the state before rotation.

[21] Interpretations such as Bohmian mechanics do not agree on this.

[22] This particular mixed state can be calculated by taking the trace of the two-particle density operator over B states: $\varrho_A = \mathrm{tr}_B\, |\psi\rangle \langle\psi|$.

$$|\psi\rangle \to |{\downarrow}\rangle_A \langle{\downarrow}|_A \frac{1}{\sqrt{2}} \left(|\varphi_A,{\uparrow}\rangle\,|\varphi_B,{\downarrow}\rangle - |\varphi_A,{\downarrow}\rangle\,|\varphi_B,{\uparrow}\rangle\right) = -\frac{1}{\sqrt{2}}\,|\varphi_A,{\downarrow}\rangle\,|\varphi_B,{\uparrow}\rangle \\ \to |\varphi_A,{\downarrow}\rangle\,|\varphi_B,{\uparrow}\rangle\ . \tag{13b}$$

That is, if we measured the spin to be up (down) for particle A, we know, since only the first (second) branch of the wave function is present after measurement, that particle B must have spin down (up). The particles may be arbitrarily far apart at the time of measurement, depending on the spatial parts of the wave function. (If the particles are created moving apart, then that motion will go on by momentum conservation, and the spatial parts of the wave function will correspond to wave packets that move apart.)

Now if we happen to know that what we measured was one partner of an entangled pair, then we also know the spin orientation of the second particle after the measurement, because it is oriented exactly opposite to that of the first partner. That is, our local measurement of one particle's spin is also a (nonlocal) measurement of the second particle's spin.

If only a spin measurement along the direction that we have chosen to define the spin basis of our state gave such a result, there would be a ready explanation in classical terms. Replace the particles by two classical tops spinning in opposite directions with axes parallel to the z axis. Let them move apart. Find out the spinning direction of the one close to you and you know the spinning direction of the other. The correlation of the two spinning directions was established before sending away one top, and it is just revealed by the measurement. But we cannot measure the spin of the classical tops along the x direction, because there is none, the z direction is distinguished.

However, the state in Eq. (11) is *isotropic* in spin space, i.e., if $\uparrow$ means orientation along the positive z axis, we can measure along the x axis and we will again find that the spin components of A and B along these directions are oppositely oriented. Bell has shown [5] that even this can be accounted for in a deterministic hidden-variable model, but he also has shown that if spin components along *different* axes are measured on both ends of the pair, there are correlations between the values of the measured spin components along three different orientations that cannot be explained by any local (realistic) model.

Every explanation of the quantum mechanical correlations in terms of an underlying realistic structure seems to require that the spins communicate. Since they have to do so even at space-like separation – the quantum mechanical results have been experimentally confirmed for that situation [6], it appears that the collapse of the wave function defies relativistic causality, because the measurement at the A spin that collapsed the wave function, may have happened *before* the measurement of the B spin in one frame of reference and *after* it in another.

The way special relativity escapes this refutation is subtle. In order to obtain the correlations experimentally that seem to require superluminal communication, it is mandatory to measure *both* spins (locally). If measurements are done on one side only, the existence of entanglement is not even provable, because all single-particle measurements are explicable by assuming the particle to be in the mixed state of Eq. (12).

If measurements are done on both sides, the correlations can be ascertained, but then in a frame of reference where A is measured first, the collapse of the wave function producing a definite spin state of both particles may be considered caused by the measurement on A, whereas in a frame, where B is measured first, that measurement will cause the collapse.

We may imagine a situation, where the observers move so that either of them finds in their inertial frame of reference that the other observer makes their measurement after them. This would suggest that both find an uncollapsed wave function and the quantum correlations should not be observed. Of course, they will still be verified, and the conclusion is just an illegitimate way of applying special relativity. The laws of special relativity in their simplest form[23] hold only in inertial systems – that is, to describe the collapse in special relativity, we should choose an inertial frame in which to do so – a *single* one, not *two* of them. It is possible to give special relativity a covariant form or to use general relativity directly in order to describe physics in a global frame that moves along with the observer of A in one place and that of B in another place. The description in such an intricate global frame[24] is complicated. Even the definition of simultaneity is. I do not have the ambition to try and give a general relativistic picture of the collapse here.

However, I would like to clarify another aspect that my simple examples have hidden so far. Looking at Eq. (13), we realize that the measurement of particle A's spin destroys the entanglement of this particle with the other – no matter what the outcome of the spin measurement, the final state is a product state, i.e., it is not entangled anymore. Is this always so? The answer is no, even for the locally measured particle.[25] Entanglement of the close-by object is destroyed when the measurement produces a nondegenerate state. On the other hand, if a degenerate eigenvalue is obtained in measuring some property, the resulting combined state may still be entangled, and usually is. To study an example, let us assume that we do not measure a spin component of the state given by Eq. (11) but the energy of particle A, in the absence of any magnetic field, so the two spin components give the same energy, and we have (at least) two-fold degeneracy. Let the measured energy be e_A and denote the two corresponding eigenstates of the Hamiltonian of particle A as $|e_A, \uparrow\rangle$ and $|e_A, \downarrow\rangle$, then the projector providing the measurement result is $|e_A, \uparrow\rangle \langle e_A, \uparrow| + |e_A, \downarrow\rangle \langle e_A, \downarrow|$ and we find

$$\begin{aligned} |\psi\rangle &\to \left(|e_A, \uparrow\rangle \langle e_A, \uparrow| + |e_A, \downarrow\rangle \langle e_A, \downarrow|\right) |\psi\rangle \\ &= \frac{1}{\sqrt{2}} \left(|e_A, \uparrow\rangle \langle e_A, \uparrow | \varphi_A, \uparrow\rangle |\varphi_B, \downarrow\rangle - |e_A, \downarrow\rangle \langle e_A, \downarrow | \varphi_A, \downarrow\rangle |\varphi_B, \uparrow\rangle\right) \\ &= \frac{1}{\sqrt{2}} c_A \left(|e_A, \uparrow\rangle |\varphi_B, \downarrow\rangle - |e_A, \downarrow\rangle |\varphi_B, \uparrow\rangle\right) \\ &\to \frac{1}{\sqrt{2}} \left(|e_A, \uparrow\rangle |\varphi_B, \downarrow\rangle - |e_A, \downarrow\rangle |\varphi_B, \uparrow\rangle\right) \end{aligned} \tag{14}$$

where $c_A = \langle e_A, \uparrow | \varphi_A, \uparrow\rangle = \langle e_A, \downarrow | \varphi_A, \downarrow\rangle$, and this state is still entangled.

Wave function collapse in alternative interpretations of quantum mechanics

Bohmian mechanics

When discussing the collapse of the wave function with advocates of Bohmian mechanics, one may encounter two attitudes, depending on the level of profundity of the researcher.

A superficial or naive answer to the question of the collapse would assert, after stating that every measurement can be reduced to a position measurement,[26] that the outcome of a measurement is

[23] That is the form in which the Lorentz symmetry is manifest and Einstein synchronization used to define distant simultaneity.

[24] That frame is necessarily time dependent, because different parts of it are moving in different directions.

[25] We are definitely ready to concede that if A was entangled with several particles, these may remain entangled among each other, even if A has become disentangled.

[26] Because we can always imagine the final piece of a measurement being a pointer taking a definite position. That need not however be the position of any Bohmian particle.

always determined by the position of a Bohmian particle which is located in some branch of the wave function and that all other branches, supposed to be empty, "can be thrown away" after the measurement. Clearly, this renders the result deterministic and suggests that the collapse is nothing but a convenience. Once we know what branches of the wave function are empty, we need not consider them anymore. The collapse looks harmless this way and certainly is not a dynamical process.

I consider this idea superficial, because it does not reflect on what would happen, if we forgo the convenience. If we just *can* throw away the empty branches of the wave function, i.e., if it is just a way of making life easier, this also means that we *need not* throw it away! We could keep it and that should not change any of the results. But that is not true. Consider our initial example, where a measurement of an observable A produces an eigenvalue a, meaning that the wave function, which was $|\psi\rangle$ before, becomes $|\psi_1\rangle \propto |a\rangle$ after the measurement. Standard quantum mechanics states that we get the correct statistical results for a *subsequent* measurement, if we assume the state before it to be $|\psi_1\rangle$, and predictions of Bohmian mechanics agree in every respect with those of standard quantum mechanics, if they use the same initial state.[27] But that means that it is not an *option* to throw away the empty part of the wave function, it is a *must*!

If it is *not* thrown away, the state to be taken as initial state of the second measurement would not be $|\psi_1\rangle$ but some state $\left|\tilde{\psi}_1\right\rangle$ the wave function of which is augmented by one or more "empty" branches of the original wave function. $\left|\tilde{\psi}_1\right\rangle$ would then, for example, not produce probability one for a result of a on immediately remeasuring A (because the empty branches produce probability densities, too). It would also generate the wrong statistics for the results of other measurements.

If the empty branch *is* thrown away, the initial state of the second measurement is $|\psi_1\rangle$, and that gives the right statistics. Therefore, the collapse is *necessary* in Bohmian mechanics as well. Given that the wave function is ontic for most Bohmianists, this makes the collapse dynamic and suggests that Bohmian mechanics should provide equations for this sometimes drastic change of the wave function, which it does not.

More profound thinkers will be ready to admit that the collapse is present in Bohmian mechanics just as in the standard interpretation, but will point out that Bohmian mechanics is explicitly nonlocal, so the ensuing clash with special relativity simply means that the ontology of special relativity is wrong. And they may emphasize that the collapse is deterministic in Bohmian mechanics in spite of the fact that there are no equations for it.

[27] This has been challenged in the literature [7] with statements to the effect that Bohmian mechanics would not capture higher-order correlations such as the correlations arising between identical particles, due to the effect that they obey Bose statistics. However, Ghose [7] does not appreciate that the many-particle wave function gives the *maximum possible quantum mechanical information* about a system at a given time (in standard quantum mechanics). It therefore contains *all spatial* (one-time) *correlations*. As long as Bohmian mechanics uses that wave function, it makes the same predictions as standard quantum mechanics to *all* correlation orders. The error in Ghose's paper [7] can be identified by realizing that he effectively uses *different* wave functions in generating predictions of standard quantum mechanics and predictions of Bohmian mechanics. So his point is not proved. On the contrary, there is a mathematical proof for the opposite of what the title of his paper claims. Of course, it is crucial for the agreement between Bohmian mechanics and quantum mechanics that the (initial) ensemble of Bohmian particles considered is precisely the one described by the probability density obtained from the (initial) wave function [8].

Many-worlds interpretation

What about the relative-state or many-worlds interpretation [9] of quantum mechanics? It claims to be local, to have no dynamical laws beyond those implied by the Schrödinger equation and to avoid the collapse entirely.

In a sense, these claims are true.[28] In the many-worlds interpretation, it may be argued that the wave function of the universe is the only ontic quantity that exists [10]. Everything else, in particular, the universe that is accessible to us is just derived, exists only, because it is part of the universal wave function. That wave function never collapses. What happens on measurement is that the universal wave function develops several branches, each one corresponding to a particular outcome of the measurement result, and that all of these continue to exist. Observers see only one result, because they branch along with the wave function and each branch of an observer sits only in a branch of the universal wave function corresponding to a particular result.

Note that this amounts to a *redefinition of reality*, making most of it completely inaccessible to observers who can only perceive the set of parameters of the world that were fixed results of measurement-like processes in their own branch, defined by these parameters.[29]

So it is true in the many-worlds interpretation that the universal wave function never collapses. But what is the use of such a statement? We can never use that wave function in a calculation, not only because it contains too many variables, but also because we sit only in a subbranch of it and have no idea what *all* the other subbranches look like.

What do we use in the Schrödinger equation? Relative wave functions, cut off from the branches by restricting consideration to much smaller systems even than our visible macroscopic world. We use states such as $|\psi\rangle$, states that may just describe a single atom, electron or photon. Even this relative state will branch on the first measurement made on it, if the measurement can yield more than one result. We may take along the macroscopic measuring apparatus such as the detector in Eq. (10) in the description, at the price of not having a single definite result of the measurement. As soon as we require one of the possible results to be true, we end up with a state such as $|\psi_1\rangle$. This is a new *relative* state, restricting our accessible world to the branch, in which the result considered has observable reality.

But the transition from $|\psi\rangle$ to the new reduced state $|\psi_1\rangle$, in the same Hilbert space, again corresponds to a collapse. So *relative states do collapse*, even in the Everett interpretation.

The only *usable* wave functions *are* relative states. These change via the Schrödinger equation and via collapse.

What the many-world interpretation does is to reinterpret the collapse as a change of the vantage point of the observer. This obviates the need for a dynamical description. Rather the collapse is a consequence of a change of the frame of reference,[30] to be considered kinematical rather than dynamical. This is very similar to what happens when a knowledge change leads to the collapse of a classical probability distribution. Nobody would even dream of finding a dynamical description for this kind of transition, because it so obviously is different from everything dynamical (and all dynamic changes are described by the Schrödinger equation).

[28] But of little relevance for us poor Earthlings. See the discussion.

[29] Of course, measurement-like processes must be assumed to have gone on since the big bang, if the picture is not to become too consciousness-dependent.

[30] However, a frame change that we cannot control. And we cannot go back to the previous frame.

The interpretation is also largely local. Because the basic ontology is that of the wave function, there is no space, rather we live in configuration space.[31] In configuration space, evolution by the Schrödinger equation is local. The collapse is local, too, as it is a change of the observer, entering a particular, preexisting branch, and the observer may be considered a localized entity. The violation of Bell's inequalities does not need nonlocal dynamics either, but it requires nonlocal objects. The relative states corresponding to world branches of the wave function can hardly be considered local.

Of course, one may still wonder and it is legitimate to ask how in this multiverse of worlds existing in parallel the quantum correlations between relative states arise that seem to make each of the worlds nonlocal. But a mere reference to quantum mechanical formulas seems less unsatisfactory in an interpretation that declares quantum mechanical entities the fundamental ontological building blocks of the fabric of reality than in an interpretation where particles or waves are the basic elements. Clearly, one has to relearn the concept of reality to some extent, if one subscribes to this interpretation. I do not want to argue in favor of, or against, it. I would just like to point out that an effective collapse of wave functions is present in the relative-state interpretation as in any other interpretation of quantum mechanics.

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[31] To make this work with relativity, is a different matter.