Mathematical Modeling and Numerical Simulation of Smart Structures Controlled by Piezoelectric Wafers and Fibers

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1 Introduction

Smart structures are characterized by a synergistic integration of active materials into a passive structure connected by a control system to enable an automatic adaptation to changing environmental conditions. The increasing engineering activities in the development and industrial applications of structronic (structure + electronic) systems require effective and reliable simulation and design tools in this field.

In vibration and noise control as well as in shape control piezoelectric wafers and fibers embedded in composite materials are very common smart structural components in structronic systems. The global behavior of such high integrated complex engineering smart structures can be simulated and designed with sufficient accuracy on the basis of the finite element method extended by control and optimization strategies. In piezoelectric controlled smart composite structures the coupled electro-thermo-mechanical field equations have to be solved, where the parameters of the material tensors are needed as input parameters. These macroscopic (homogenized) material tensors of such composites are non-linearly dependent on the properties as well as the arrangements of the constituents in the composite. The measurement of these data is time consuming and expensive. Alternatively, analytical (e.g. based on the Mori-Tanaka-type mean field approach) as well as numerical (e.g. based on the finite element analysis of a representative volume element) methods can be used to calculate homogenized material tensors of a heterogeneous material. The paper first presents the application of the Effective Field Approximation (EFA) for the analytical determination of homogenized material tensors of electro-thermo-mechanical material systems, consisting of a matrix material with piezoelectric inclusions (e.g. in form of thin piezoelectric fibers). Then a brief description of a general purpose finite element package for the simulation of electro-thermo-mechanical coupled field problems is given. The presented analytical homogenization strategy can be simply incorporated in the finite element simulation tool. Finally, as a simple test example an actively controlled beam structure is given, where piezoelectric fiber wafers are used as actuators. The homogenized material tensors are calculated first based on the EFA. These data are than used in the finite element simulation.
2 Constitutive equations

The general linear relationship between the field quantities stress $\sigma$, strain $\gamma$, electric field $E$ and dielectric displacement $D$ are written here in the form

\[
\begin{align*}
\sigma_{ik} &= \epsilon_{ikm} \gamma_{lm} - \epsilon_{ik} E_l - \lambda_{ik}^0 \theta \\
D_j &= \epsilon_{jik} \gamma_{lm} + \epsilon_{ij} E_l + \rho_j^0 \theta
\end{align*}
\]

(1)

where we have introduced the standard notation for the linear material property tensors ($\theta$ denotes the temperature change). These constitutive equations can be written in the compact form

\[
\begin{align*}
f_p &= L_{pq} f_q + f_p^p \\
p, q &= 1, 2, \ldots, 9
\end{align*}
\]

(2)

with the $9 \times 9$ material property matrix

\[
L = \begin{pmatrix}
\epsilon_{iik} & \epsilon_{ij} \\
\epsilon_{ik} & -\epsilon_{ij}^0
\end{pmatrix}
\]

(3)

This type of constitutive equations can be applied to describe the material behavior of single components as well as of piezoelectric composites.

3 Homogenized properties of composites

In order to derive homogenized properties of fiber reinforced materials, we apply here the effective field method. This method is especially useful if we are concerned with a matrix material $L^{(0)}$ which contains inclusions $L^{(k)}$ ($k = 1, 2, \ldots, n$). In the theoretical scheme every inclusion is embedded in an infinite matrix medium, and the interaction between different inclusions is projected into an effective field which acts on the considered inclusion. This effective field is then determined in a self-consistent manner$^{5,6,7}$. The result for the effective (homogenized) material properties (here denoted by an upper asterisk) can be written into the following concise form

\[
\begin{align*}
\mathbf{L}^{(1)} &= \left( A(L, L^{(0)}) \right)^{-1} \left( A(L, L^{(0)}) \right) \\
\mathbf{f}^{(1)} &= \left( A(L, L^{(0)}) \right)^{-1} \left( A(L, L^{(0)}) \mathbf{f}^{(0)} \right)
\end{align*}
\]

(4)

(5)

where the $9 \times 9$ matrix

\[
A(L, L^{(0)}) = \left( I + (L - L^{(0)}) P^0 \right)^{-1}
\]

(6)

depends on the matrix properties $L^{(0)}$, the inclusion properties $L$, and on $P^0$ which is the generalized Eshelby Tensor formed with the matrix properties (see e.g. Huang and Kuo$^5$). This Eshelby tensor depends also on the shape of inclusions which are approximated here as spheroids, where the aspect ratio is used to model different inclusion geometry. The angular brackets denote averaging over all types of components.
4 Finite element analysis for coupled problems

The basic equation for the finite element analysis of an coupled electro-mechanical field problem can be obtained as virtual energy formulation that combines the Cauchy’s equation of motion

\[
\sigma_{ij,j} + \rho B_i = \rho \dot{u}_i
\]

(7)

with the electrical balance equation

\[
D_{ij,j} = 0
\]

(8)

in form of

\[
G = \int (\sigma_{ij,i} + \rho B_i - \rho \dot{u}_i) \delta \phi_j dV + \int (D_{ij,j}) \delta \phi_j dV = 0.
\]

(9)

With the constitutive equations (1), the strain displacement relation \( \gamma_{ij} = \frac{1}{2}(\sigma_{ij} + \sigma_{ji}) \) and the relation between the electric field and the electric potential \( E_i = \phi_i \) we can express equation (9) by the unknown field variables \( u_i \) and \( \phi \). We approximate these field variables in a finite element by shape functions

\[
u_i = \sum_{(L)} N_u^{(u)}(u) u_L, \quad \phi = \sum_{(L)} N_{\phi}^{(\phi)} \phi_L
\]

(10)

where \( N_u^{(u)} \) and \( N_{\phi}^{(\phi)} \) are the shape function for the displacements and the electric potential, respectively. Following the standard procedure for the development of finite element equations we get the semi-discrete form of the equations of motion for a coupled electro-mechanical problem in the form

\[
\begin{bmatrix}
M_{uu} & 0 & \dot{u} \\
0 & 0 & u
\end{bmatrix} + \begin{bmatrix}
R_u & 0 & \dot{\phi} \\
0 & 0 & \phi
\end{bmatrix} + \begin{bmatrix}
\omega & 0 \\
0 & -\omega & 0
\end{bmatrix} \begin{bmatrix}
\phi \\
\phi
\end{bmatrix} = \begin{bmatrix}
f_u \\
f_{\phi}
\end{bmatrix}
\]

(11)

Based on the above given theoretical background, a library of piezoelectric finite elements has been developed and tested. This library includes solid elements, plane elements, axisymmetric elements, rod elements as well as special multilayer composite shell elements and has been implemented in the general purpose finite element system COSAR. For static solutions a specially optimized sub-matrix oriented Cholesky solver can be used. For the solution of eigenvalue problems the subspace iteration method by Mc Cormick&Noe is used. In transient problems modal based techniques as well as time integration schemes such as the Newmark-Method, the Wilson-Method, or the Central-Difference-Method can be applied. The finite element code has the capability to use a substructure technique. This can be utilized as an excellent precondition to solve optimization problems and nonlinear problems effectively. The finite element software can also be used to simulate controlled structures. A data interface between the finite element software and control design tools such as MATLAB/SIMULINK has been developed to design controller for real engineering applications.
5 Numerical results

First, homogenized material properties of composites made of piezoelectric fibers embedded in a polymer matrix are calculated on the basis of the effective field method presented. The electromechanical material properties of each of the two constituents of the composite as well as the calculated homogenized data for a special composite consisting of a 30% fiber volume fraction and a fiber aspect ratio of 50 are given in the Table 1.

<table>
<thead>
<tr>
<th></th>
<th>$c_{11}^h$</th>
<th>$c_{12}^h$</th>
<th>$c_{13}^h$</th>
<th>$c_{33}^h$</th>
<th>$c_{44}^h$</th>
<th>$e_{31}^h$</th>
<th>$e_{33}^h$</th>
<th>$e_{15}^h$</th>
<th>$\varepsilon_{11}^h$</th>
<th>$\varepsilon_{33}^h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>PZT-fibers</td>
<td>106.0</td>
<td>58.2</td>
<td>57.4</td>
<td>90.2</td>
<td>20.0</td>
<td>-4.1</td>
<td>12.1</td>
<td>8.6</td>
<td>14.0</td>
<td>13.0</td>
</tr>
<tr>
<td>Polymer</td>
<td>6.4</td>
<td>3.5</td>
<td>3.5</td>
<td>6.4</td>
<td>1.5</td>
<td>0</td>
<td>0</td>
<td>0.04</td>
<td>0.04</td>
<td>0.04</td>
</tr>
<tr>
<td>Composite</td>
<td>9.8</td>
<td>5.0</td>
<td>5.2</td>
<td>22.1</td>
<td>2.6</td>
<td>-0.1</td>
<td>3.3</td>
<td>0.0</td>
<td>0.08</td>
<td>3.1</td>
</tr>
</tbody>
</table>

In Figures 2a and 2b results are presented for different volume fractions and fiber aspect ratios.

![Figure 1](image-url)

**Figure 1.** Predicted piezoelectric constants $e_{33}$ (a) and $d_{33}$ (b) for a composite made of piezoelectric fibers in a polymer matrix
aspect ratios which show that a high fiber volume fraction and a large aspect ratio is required to get the desired performance of the composite. This is especially important when the $e_{33}$-effect has to be utilized, but is not so critical for the $d_{33}$-effect. Note that all calculations have been performed with an identical poling state of the fibers as shown in Table 1. If the poling of the composite is done "in-situ", the resulting poled fiber properties may vary in the same manner as the effective properties of the composite, which can be also calculated within the same theoretical approximation scheme. The main advantages of fiber composites are the better structural conformity in comparison with PZT wafers (see the low stiffness properties in Table 1) as well as the an-isotropic sensing and actuation behavior.

The homogenized material properties were used for a numerical simulation. As test case a beam is used with two attached piezoelectric fiber composite layers at top and bottom of the beam (see Figure 2), where these layers consist of the homogenized material properties given in Table 1. The actuators are controlled with a time depending electric potential $\phi(t) = \phi_v \sin(\Omega t)$ with $\phi_v = 100\text{V}$, $\Omega = 100\text{s}^{-1}$. In Figure 3 the displacement response of a point at the free end of the beam is shown.

![Figure 2. Elastic beam controlled by piezoelectric composite fiber patches](image)

**Figure 2.** Elastic beam controlled by piezoelectric composite fiber patches

![Figure 3. Response of the beam in the time domain](image)

**Figure 3.** Response of the beam in the time domain
6 Conclusions

A simulation concept for piezoelectric controlled smart structures has been presented, where in the first step a homogenization method based on the Effective Field Approximation is given. In the second step the homogenized material tensors are used in the finite element simulation of the global structural behavior. In the near future this solution will be used to optimize smart material systems with respect to the overall performance of structronic systems.

Acknowledgment

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References

1. H. S. Tzou, G. L. Anderson (Eds.), Intelligent Structural Systems, Kluwer 1993