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Accurate Modeling of the Electric Field within Piezoelectric Layers for Active Composite Structures

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ABSTRACT: The article considers thin-walled active structures, which utilize the piezoelectric patches as both sensor and actuator components. Most of the developed models for this type of application make an assumption of a constant electric field and, consequently, a linear distribution of the electric potential over the thickness of the piezopatches. Some recent papers use higher-order functions to model the mentioned electric quantities. In the study, it is demonstrated through an analytical deduction that a quadratic distribution of the electric potential and a linear distribution of the electric field are adequate for the piezoelectric patch that exhibits kinematics described by a first-order two-dimensional theory. A degenerated shell element is developed for modeling purposes and a set of numerical analyses is performed in order to demonstrate the additional stiffening effect caused by the refined functions for the electric quantities. The significance of the effect is discussed in detail.

Key Words: active structure, piezoelectric composites, finite shell element.

INTRODUCTION

Science and technology have made amazing developments in the design of structures using classical materials. Nevertheless, structures made of classical materials are passive systems dimensioned so as to be capable of undergoing the most critical loadings that occur quite rarely or possibly even never during the lifetime of the structure. This actually means that they are oversized for most of the operation, which has a significant influence on their production price and operational costs. Furthermore, this passiveness influences the robustness of the system – any critical, but unforeseen, loading is a potential danger and, additionally, there is no possibility of active improvement of the behavior of the structure during the act of the loads. The idea on how to cope with those drawbacks of passive systems introduces terms such as active or smart materials and systems. Generally speaking, active or adaptive structure is a structure, typically fabricated from composite materials, that includes bonded or embedded smart-material-based sensors and actuators with a control system that enables the structure to respond in real time or nearly real time to external stimuli to compensate for undesired behavior or to produce a desired response through the change of the structure’s stiffness, inertial properties, damping properties, or configuration.

In recent years the study of active structures has attracted many researchers because of their potential benefits in a wide range of applications, such as shape control, vibration suppression, noise attenuation, damage detection, etc. (Gandhi and Thompson, 1992; Gabbert, 2002). Piezoelectric materials belong to the group of most frequently used smart materials, especially when thin-walled structures are treated with the aim of vibration suppression. They are appropriately chosen with respect to the required level and frequency band of actuating and sensing loads and embedding requirements. Hence, the analysis of thin-walled structures with embedded active piezoelectric elements is of increasing importance, whereby an accurate modeling of the mechanical and the electric field are required in order to capture correctly the piezoelectric coupling. Nowadays computationally efficient formulations are by default addressed to the finite element method. The calculation of coupled electromechanical systems in commercially available FE-software packages can be conducted only with a quite limited number of finite elements. For example, up to now ANSYS and ABAQUS offer only three-dimensional (3D) elements that take into account the piezoelectric effect (Mesecke-Rischmann, 2004). The application of these elements for the analysis of thin-walled structures leads to a significant numerical effort. Therefore, numerous
researchers in this field have devoted their work to the development of more adequate 2D finite elements that offer a satisfactory accuracy with less numerical effort. The study represents such a model with special attention focused on the accurate modeling of the electric field in the piezoelectric actuators and sensors. The aim is to investigate its significance for the prediction of the general behavior of the active structure and to identify the parameters influencing the mentioned significance.

**KINEMATICAL ASSUMPTIONS**

Thin-walled structures made of a multilayered material involving piezoelectric active components are considered in the study. Hence, 2D theories are appropriate for the purpose of modeling those structures. Most of the real-world problems, especially involving composite structures, do not admit exact solutions, requiring one to find an approximate but representative solution. Also, one should be aware of the objectives of the modeling as well as the 'numerical effort–achieved accuracy' ratio. If the prior objective is to get a general behavior of the structure, without analyzing the local effects, i.e., effects in a micro-domain, then a 2D theory of the first order is quite satisfactory in most cases.

The model is supposed to cover thin as well as moderately thick structures, and therefore First-order Shear Deformation Theory (FSDT) is used. The theory is based on the Mindlin–Reissner kinematical assumptions (Ochoa and Reddy, 1992): a straight line originally perpendicular to the mid-surface remains straight and inextensible after deformation, but does not remain necessarily perpendicular to the mid-surface any more. Hence, the displacement field is described in the local coordinate system (denoted as \( x, y, z \)) in Figure 1 in terms of degrees of freedom of the mid-surface point as:

\[
\begin{align*}
    u &= u_0 + z\theta_x \\
    v &= v_0 - z\theta_y \\
    w &= w_0,
\end{align*}
\]

where \( u_0 \) and \( v_0 \) are in-plane displacements, \( w_0 \) is the transverse deflection, and \( \theta_x \) and \( \theta_y \) are rotations of the transverse normal around the \( x- \) and \( y- \) axis, respectively, all of them at the mid-surface point and they are functions of the mid-surface point position. A linear case is assumed, i.e., small strains and small displacements, so the strain field is defined by kinematical relations from the displacement field in the following way:

\[
\begin{align*}
    \varepsilon_{11} &= \frac{\partial u_0}{\partial x} + z\frac{\partial \theta_x}{\partial x} = \varepsilon_{11}^0 + z\kappa_{11}^f; \\
    \varepsilon_{22} &= \frac{\partial v_0}{\partial y} - z\frac{\partial \theta_x}{\partial y} = \varepsilon_{22}^0 + z\kappa_{22}^f; \\
    \gamma_{12} &= \left( \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \right) + z\left( \frac{\partial \theta_x}{\partial y} - \frac{\partial \theta_y}{\partial x} \right) = \gamma_{12}^0 + z\kappa_{12}^f; \\
    \gamma_{23} &= \frac{\partial w_0}{\partial y} - \theta_x,
\end{align*}
\]

where a distinction between the membrane strains \( \varepsilon_m = [\varepsilon_{11}^0, \varepsilon_{22}^0, \gamma_{12}^0]^T \), the flexural strains \( z[\kappa_f]^T = [\kappa_{11}^f, \kappa_{22}^f, \kappa_{12}^f]^T \), and the transverse shear strains \( \{\gamma_s\}^T = [\gamma_{23}] \) can be made, with \( \kappa_f \) denoting the curvatures of deformation.

Having defined the basic assumptions pertaining to the displacement field, we can proceed by considering the piezoelectric continuum exhibiting such kinematics.

**PIEZOELECTRIC PATCH POLARIZED IN THICKNESS DIRECTION**

The form of the linear piezoelectric constitutive equations depends on the choice of independent variables (Ikeda, 1996). Since the model is aimed at the structural analysis and the vibration suppression, it is suitable to choose the mechanical strain and the electric field as independent variables (Rosen et al., 1992), yielding:

\[
\begin{align*}
    \{\sigma\} &= [e^T][e] - [e]^T[E] \quad (3) \\
    \{D\} &= [e][\varepsilon] + [d][E] \quad (4)
\end{align*}
\]

where \( \{\sigma\} \) and \( \{\varepsilon\} \) represent the mechanical stress and strain in the vector notation, respectively; \( \{D\} \) and \( \{E\} \) are the electric displacement and the electric field vector, respectively; the matrix \( [e] \) comprises the piezoelectric coupling constants; and \( [d] \) is the matrix of dielectric constants.

---

**Figure 1.** Generally shaped thin-walled structure and Mindlin–Reissner kinematics in the local coordinate system.
Commercially available piezoelectric ceramics possess the material properties of a transversely isotropic material, whereby the axis normal to the plane of isotropy is in the thickness direction (denoted by subscripting \( 3 \)). Hence, the above equations can be given together in a developed form as follows:

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{12} \\
\sigma_{13} \\
\sigma_{23} \\
D_1 \\
D_2 \\
D_3
\end{bmatrix} =
\begin{bmatrix}
Q_{11} & Q_{12} & 0 & 0 & 0 & e'_3 \\
Q_{12} & Q_{11} & 0 & 0 & 0 & e'_3 \\
0 & 0 & Q_{66} & 0 & 0 & 0 \\
0 & 0 & 0 & Q_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & Q_{44} & 0 \\
e'_1 & e'_2 & e'_3 & 0 & 0 & -d_{33}
\end{bmatrix}
\]

where \( e'_3 \) is the electric displacement in the thickness direction, \( x_3 \), and the thickness direction of the piezoelectric patch coincides with the thickness direction of the structure to which the patch is attached, or embedded, denoted by \( z \). Nevertheless, the origins of the two axes do not coincide in general (Figure 1).

Now, expressing the electric displacement \( D_3 \) from Equation (6) and introducing it into Equation (9), it becomes:

\[
\frac{\partial D_3}{\partial x_3} = e'_3 \left( \frac{\partial e'_{11}}{\partial x_3} + \frac{\partial e'_{22}}{\partial x_3} \right) + d_{33} \frac{\partial E_3}{\partial x_3} = 0. \tag{10}
\]

Equation (2) yields that the partial derivatives of the membrane and flexural strains with respect to the thickness coordinate (\( z \)) are the curvatures of deformation, and they are constant with respect to the thickness coordinate (but functions of the in-plane coordinates). Taking into account that \( E_3 = -\frac{\partial \varphi}{\partial x_3} \), where \( \varphi \) denotes the electric potential in the thickness direction (the index for the electric potential is omitted), from Equation (10) we have:

\[
\frac{\partial^2 \varphi}{\partial x_3^2} = \frac{e'_3}{d_{33}} (\kappa_1(x_1, x_2) + \kappa_2(x_1, x_2)) = k_p(x_1, x_2). \tag{11}
\]

Hence, the second derivative of the electric potential with respect to thickness coordinate is constant over the thickness, and it is only a function of in-plane coordinates \( x_1 \) and \( x_2 \), which results in the following general function for the electric potential:

\[
\varphi(x_1, x_2, x_3) = \frac{1}{2} k_p(x_1, x_2) x_3^2 + b(x_1, x_2) x_3 + c(x_1, x_2). \tag{12}
\]

So, the electric potential is described as a quadratic function in the thickness direction, which shows that the usual approximation that yields a linear function for the electric potential does not correspond to the kinematical assumptions, resulting in linear strain distribution. The integration constants \( b \) and \( c \) are to be determined upon imposing boundary conditions. It should be noted that only the electric potential difference between the electrodes of the piezoelectric patch, \( \Delta \varphi \), is of importance.
Therefore, it can be assumed that the electric potential of the lower electrode is zero and that of the upper is equal to \( \Delta \varphi \) (Figure 2(b)).

Denoting the thickness of the piezoelectric patch by \( h_p \), and placing the origin of the coordinate system in the mid-surface, the lower surface will be defined by \( x_3 = -h_p/2 \), and the upper by \( x_3 = h_p/2 \), so that:

\[
\varphi(x_1, x_2, -h_p/2) = 0 \quad \text{and} \quad \varphi(x_1, x_2, h_p/2) = \Delta \varphi. \tag{13}
\]

It is now a straightforward task to determine the integration constants, and the final expression for the electric potential is obtained as:

\[
\varphi(x_1, x_2, x_3) = \frac{1}{2} k_x(x_1, x_2) \left( x_3^2 - \frac{h_p^2}{4} \right) + \Delta \varphi \left( \frac{x_3}{h_p} + \frac{1}{2} \right).
\tag{14}
\]

The electric field distribution can be easily determined as:

\[
E(x_1, x_2, x_3) = -\frac{\partial \varphi(x_1, x_2, x_3)}{\partial x_3} = -k_x(x_1, x_2) x_3 - \frac{\Delta \varphi}{h_p}.
\tag{15}
\]

Thus, corresponding to a linear distribution of mechanical strains the electric field is a linear function with respect to the thickness coordinate, and is not a constant one as it is usually assumed. The constant term in the linear electric field function is the same as the usual approximation for the electric field (\( E_z = -\Delta \varphi/h_p \)). As a matter of fact, it can be recognized that the common constant approximation of the electric field is a smoothed function of the actual linear distribution. The smoothed function for the electric field provides that the Gauss law is fulfilled in a weak manner. On the other hand, this formulation with the polynomials of higher order fulfills the Gauss law exactly. Furthermore, the highest-order terms are caused by bending of the patch. If the piezopatch is not bent, but only stretched, then the curvature of deformation is zero, and consequently, the linear term in the electric field function, as well as the quadratic term in the electric potential function, is equal to zero (\( k_x = 0 \)), resulting in the above-mentioned usual approximations.

As has been already pointed out, the \( x_3 \)-coordinate is the local thickness coordinate of the piezoelectric patch. The patch, regardless of whether it plays the role of an actuator or a sensor, represents an additional layer in the sequence of layers, and generally, it can take any place in the sequence. The same equations for the electric potential and the electric field are valid in the local coordinate system of the structure \((x, y, z)\), only the translation of the coordinate system in the thickness direction (Figure 2(a)) should be taken into account if the quantities are to be expressed within the structure reference system. Assuming that the piezoelectric layer is the \( k \)-th layer in the sequence, it will be:

\[
x_3 = z - \frac{z_{k-1} + z_k}{2},
\tag{16}
\]

where \( z_{k-1} \) and \( z_k \) are the distances of the lower and the upper surface of the \( k \)-th layer from the mid-surface of the structure. This relation is to be included in the expressions for the electric potential (Equation (14)) and the electric field (Equation (15)).

**FINITE ELEMENT FORMULATION OF THE PROBLEM**

The governing equation of the structure’s dynamical behavior is given by the Hamilton’s principle, i.e., the system takes the path of the stationary action:

\[
\delta \int_{t_1}^{t_0} (L + W) dt = 0
\tag{17}
\]

where \( L \) represents the Lagrangian, and \( W \) is the virtual work of external forces. The Lagrangian is to be properly adapted in order to include the contribution from the electrical field besides the contribution from the mechanical field. Since the mechanical strain and the electric field are independent variables, the governing thermodynamic equation for the piezoelectric material is the electric enthalpy (electric Gibbs energy), \( H \) (Ikeda, 1996):

\[
H = \frac{1}{2} c_{ijkl} e_{ij} e_{kl} - e_{mn} e_{mn} - \frac{1}{2} d_{mn} E_m E_n
= \frac{1}{2} \left[ (e)^T (e) - (E)^T (D)_G \right]
\tag{18}
\]
Modeling of Electric Field within Piezoelectric Layers

and the Lagrangian is defined in terms of mechanical kinetic energy $E_{\text{kin}}$ and electric enthalpy as $L = E_{\text{kin}} - H$. Upon integration by parts of the kinetic energy, the Hamilton’s principle for the piezoelectric continuum can be rewritten in the developed form:

$$
\int_V \left[ \rho \dot{\delta u}^T \dddot{u} + (\delta \varepsilon)^T [C^E] \dot{\varepsilon} - (\delta \varepsilon)^T [C] \delta \varepsilon \right] dV - \int_V \delta \varepsilon^T \dddot{u} dV - \int_V (\delta E)^T \delta u^T dV = \int_S \overline{\delta F}_v dS_v + \int_S \overline{\delta F}_p dS_p - \int \ddot{\delta \phi} q dS_2 - \delta \phi Q
$$

where $F_v$, $F_{Sl}$, and $F_p$ are the external volume loads, surface loads (acting on surface $S_l$) and point loads, respectively, and $q$ and $Q$ are the surface electric charge (acting on surface $S_2$) and the point electric charges, respectively. Due to the assumption of a constant difference of the electric potential over the surface of each piezoelectric layer, only the corresponding electrical loads are considered and those are the uniformly distributed surface electric charges.

The strains are obtained by means of the derivation operator matrix $[D]$, so that $\{ \varepsilon \} = [D] \{ u \}$. The discretization of the structure and the displacement field is a function of both the strain field and the field is a function of both the strain field and the electric enthalpy $E_{\text{cont}}$ and the dielectric stiffness matrix $[K_{\varepsilon}]$ yields the strain–displacement matrix $[N_{pe}] = [D]^{-1} [B]$. For a shell type of a structure modeled by a first-order 2D theory, the strain field can be represented in the natural coordinate system $(r, s, t)$ (Marinković and Gabbert, 2004; Marinković et al., 2004) as:

$$
\{ \varepsilon \} = [B] \{ u \} = \sum_{i=1}^{q} \left[ \begin{array}{c} [B_{TM}] \ \{ \delta \varepsilon \} \\
[B_{TS}] \ + \ [B_{RR}] \end{array} \right] \left[ \begin{array}{c} \{ u \} \\
\{ \delta u \} \end{array} \right]
$$

where $\{ u \}$ is the vector of all nodal mechanical degrees of freedom of the element in the global coordinate system, the $B$-matrices denoted by ‘$m$’, ‘$r$’, and ‘$s$’ in the subscript define the membrane, flexural, and shear strains, respectively, with those having subscript ‘$T$’ are related to the translations and those with ‘$R$’ are related to the rotations, and, finally, ‘0’ denotes constant terms while ‘1’ denotes linear terms with respect to the natural thickness coordinate $t$; $\{ u_{TM} \}$ are nodal translations, and $\{ u_{TR} \}$ nodal rotations of the $i$th node in the global coordinate system. The $t$-coordinate is the thickness coordinate in the natural coordinate system ($-1 < t < +1$), and it is related to the $z$-coordinate by $t = \frac{z}{h}$.

From Equation (15) it can be noticed that the electric field is a function of both the strain field and the differences of the electric potential. An active structure may comprise more than one piezoelectric layer, generally $N_p$, of them. Therefore, electrical quantities are observed layerwise, and in the finite element model they are given in the condensed form of vectors $\{ E_t \}$ and $\{ \phi_t \}$ defined on the element level. The former comprises electric field distributions in the thickness directions, while the latter comprises electrical degrees of freedom, i.e., the differences of electric potentials of all piezoelectric layers across the thickness of the element. Thus, the dimension of the both vectors is $N_p$, and after the discretization of the structure, one can write:

$$
\{ E_t \} = -\frac{\partial}{\partial t} [B_{\varepsilon}^T] \{ u_t \} - [B_{\phi}] \{ \phi_t \}
$$

where $[B_{\varepsilon}]$ defines the part of the strain field, whose partial derivatives with respect to natural thickness coordinate $t$ contribute to the electric field and $[B_{\phi}]$ is the electric field–electric potential matrix, which has a diagonal form since the difference of electric potentials of a piezoelectric layer affects the electric field only of the very same layer.

The discretization of the continuum results in the following system of semi-discrete equations:

$$
[M_{\varepsilon}] \{ u_t \} + [K_{\varepsilon}] \{ u_t \} + [K_{\phi}] \{ \phi_t \} = \{ f_{\text{ext}} \}
$$

$$
[K_{\varepsilon}] \{ u_t \} + [K_{\phi}] \{ \phi_t \} = \{ q_{\text{ext}} \}
$$

where the following matrices and vectors are defined on the element level:

- mass matrix: $[M_{\varepsilon}] = \int_V [N_{pe}]^T \rho [N_{pe}] dV$
- mechanical stiffness matrix: $[K_{\varepsilon}] = \int_V \left( [B_{\varepsilon}]^T [C^E] [B_{\varepsilon}] + [B_{\varepsilon}]^T [C] \frac{\partial}{\partial t} [B_{\varepsilon}] \right) dV$
- piezoelectric coupling matrix: $[K_{\varepsilon\phi}] = \int_V \left( [B_{\varepsilon}]^T [C^E] [B_{\phi}] + [B_{\phi}]^T [C] \frac{\partial}{\partial t} [B_{\varepsilon}] \right) dV$
- dielectric stiffness matrix: $[K_{\phi\phi}] = \int_V [B_{\phi}]^T [C] [B_{\phi}] dV$
- mechanical loads: $\{ f_{\text{ext}} \} = \int_V [N_{pe}]^T [F_{v}] dV + \int_S [N_{pe}]^T [F_{s}] dS + [N_{pe}]^T [F_{p}]$
- electric charge: $\{ q_{\text{ext}} \} = -\int_S q dS_2

The remaining terms after the first one on the right-hand side of Equation (25) represent the already mentioned additional mechanical stiffening effect, and our attention will be focused on it. In the case of a transversely isotropic piezoelectric material, it can be shown that all three terms are actually the same matrices. Performing the integration in the natural coordinate system, after the analytical integration in the thickness direction, we obtain the matrix describing the additional stiffening effect as:

\[
[K_{\text{mech}}]_{ij} = \frac{2}{3} \sum_{k=1}^{N_{pe}} c_{ijk} c_{pmk} \times \int_{-1}^{1} \int_{-1}^{1} \begin{bmatrix} [0] & [0] & \cdots & [0] \\ [0] & [B_{Rij}] & \cdots & [B_{Rij}] \\ \cdots & \cdots & \cdots & \cdots \\ [0] & \cdots & \cdots & [B_{Rij}] \end{bmatrix} \times \det(J) \, d\gamma 
\]

where the summation is performed over all \(N_{pe}\) piezoelectric layers, and \(c_{ijk}\) and \(c_{pmk}\) are the thickness geometrical factor and the piezoelectric material factor, respectively, of the \(k\)th piezolayer:

\[
c_{ijk} = \left( \frac{h_{pk}}{h} \right)^3 
\]

\[
c_{pmk} = \frac{c_{31}^2}{d_{33}^2} 
\]

with \(h_{pk}\) denoting the thickness of the \(k\)th piezolayer and \(h\) is the overall laminate thickness. \([J]\) in the Equation (30) is the Jacobian, \(i\) and \(j\) represent related element nodes, and the matrix \([B_{R}]\) is the Boolean matrix of the following form:

\[
[B_{R}] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

Thus, Equation (30) claims that the additional mechanical stiffening effect pertains only to the rotational degrees of freedom, i.e., to the bending modes. The question arises: how significant is the additional stiffening effect? An intuitive answer can be given very easily. There is a great number of formulations, which do not take this effect into account but still yield very satisfactory results. This fact already answers the question, and a small analysis of the Equation (30) explains it. Actually, four different parameters influencing the importance of the stiffening effect can be distinguished:

1. The thickness geometrical factor \(c_{ijk}\): The additional stiffening effect depends on the third power of piezolayer thickness to overall thickness ratio, (Equation (31)). The thickness of the piezopatches is severely limited due to the pronounced brittleness of the piezoelectric ceramics.

2. The piezoelectric material factor \(c_{pmk}\): The higher the values of the piezoelectric coupling constants and the lower the values of the dielectric constants, the greater the significance of the stiffening effect (Equation (32)). Furthermore, the dependence on the piezoelectric coupling constant is quadratic.

3. The surface covered by the piezolayers: The integration in Equation (30) is performed only over the part of the structure surface covered by piezopatches. If this surface represents a relatively great part of the structure undergoing a deformation, then the influence of the stiffening effect will be higher and vice versa. Usually quite a small surface of the structure is covered by active elements.

4. The 'genuine' mechanical stiffness of the material: The integration in Equation (30) involves only piezolayers. If the 'genuine' mechanical stiffness of both passive and active (piezoelectric) material resulting from their mechanical properties is too high in comparison to the additional stiffening effect, then the effect will be practically unnoticeable, and this is often the case. It should be noted that this parameter depends on the mechanical properties and thickness of both active and passive layers, the stacking sequence of layers, boundary conditions, etc.

It should be pointed out that the introduced piezoelectric material factor \(c_{pm}\) does not change drastically for different materials from the group of commercially available piezoelectric ceramics. However, the group of piezoelectric polymers has a significantly different piezoelectric coupling and dielectric constants, yielding quite different values for the piezoelectric material factor.

**NUMERICAL EXAMPLES**

The authors of this article have developed a 9-node degenerated shell element capable of modeling arbitrarily shaped thin-walled active structures made of composite multi-layered material (Marinković et al., 2004). The element is based on the first-order shear deformation theory and is capable of taking into account the additional stiffening effect due to the more accurate linear distribution of the electric field over the thickness of the piezoelectric layers. The element is implemented in COSAR, a general purpose finite element package developed at the Institute of Mechanics, Otto-von-Guericke University of Magdeburg.

A set of numerical examples is chosen to demonstrate the influence of the above-mentioned parameters and...
to make appropriate conclusions about the additional stiffening effect. Only static cases will be considered in this article, but one should be aware of the fact that any additional stiffness could play a more important role in a dynamic case exciting higher eigenmodes. Three different examples are chosen, in each of which the structure is actuated by a pair of inversely polarized piezoelectric patches.

The first example: A piezoelectric bimorph pointer is considered in the first example. The example is originally proposed by Hwang and Park (1993) and is accepted as a benchmark problem (e.g., Tzou, 1993; Gabbert et al., 1998). The device is used for micro-actuation or strain sensing. The originally proposed setup is a bimorph beam made of polyvinylidene fluoride (PVDF), the length of which is $L = 100$ mm, the width $B = 5$ mm, and the thickness $h = 1$ mm (Figure 3). The voltage of $\Delta \Phi = 1$ V is applied across the thickness of the beam along the whole length.

In this case, the piezoelectric material covers the whole thickness of the structure, which corresponds to a case with the highest values of the thickness geometrical factor $e_{1g}$. Actually, the entire structure is made exclusively of piezoelectric material. The deflection of the top of the beam is given by the theoretical expression in Figure 4 and it is $3.45 \times 10^{-7}$ m. The theoretical consideration assumes a constant value of the electric field across the thickness. The corresponding values, i.e., without the additional stiffening effect, are calculated by means of the 3D hexahedron element ($3.422 \times 10^{-7}$ m, the curve not given in the figure for the sake of a better layout) and with the degenerated shell element ($3.450 \times 10^{-7}$ m), whereby the latter needs only one element to yield the theoretical result. The degenerated shell element was also used to give the result with the additional stiffness, which is in this case $3.42 \times 10^{-7}$ m. A difference of 0.87% when the additional stiffness is included can be noticed. The obtained results are seen as nearly overlapping in Figure 4 due to quite small differences.

Now, in order to demonstrate the influence of the piezoelectric material factor, the same structure, only made of PIC151 (properties given in Figure 6) and exposed to exactly the same conditions, is considered. PIC151 has higher values of piezoelectric coupling constants, and therefore the stiffening effect is much higher (Figure 5). The deflection of the top of the beam if the effect is not taken into account yields $24.244 \times 10^{-7}$ m, and with the effect accounted for, it is $22.282 \times 10^{-7}$ m. A much higher difference of approximately 8% can be noticed in this case.

The second example: In the second example, two piezopatches are bonded to the surfaces of a clamped thin beam and a voltage of 100 V is applied to both of them. Due to the opposite polarization of the patches, their activation produces internal bending moments uniformly distributed over the edges of the patches. The piezoceramic patches have a thickness of 0.2 mm and are made of PIC151 material, while the beam is made of aluminum and has a thickness of 0.5 mm. Hence, the thickness geometrical factor is much smaller in this case, but is still relatively large in comparison to most of the similar applications. It should be noticed that the whole part of the structure undergoing deformation is covered by active layers. The remaining part of the structure either performs no motion at all (the part between the constraint and patches), or performs a rigid body motion (the part between the patches and free edge of the beam), (Figure 6).

It can be said that aluminum belongs to the group of average structural materials with respect to mechanical stiffness.

Figure 7 represents the static deflection along the beam length (centerline). As can be noticed, the difference between two formulations is small and the results are once again seen as overlapping in the figure. The calculated deflection of the top of the beam when the stiffening effect is not taken into account is $-0.605$ mm, and with the stiffening effect, it is $-0.5988$ mm. Thus, a difference of approximately 1% is noticeable. The 3D hexahedron element with a finer mesh than the one used by the degenerated shell element gave the result of $-0.610$ mm, whereby the stiffening effect is not included. Similarly to the previous case,
Figure 4. Deflection of the PDVF piezoelectric bimorph beam.

Figure 5. Deflection of the PIC151 piezoelectric bimorph beam.

Figure 6. Model of a clamped beam with piezoelectric actuators and material properties.

Passive layer – Aluminum: Young’s modulus $Y = 7.03 \times 10^4$ N/mm$^2$
Poisson’s ratio $\nu = 0.345$
Density $\rho = 2.69$ g/cm$^3$

Piezoelectric layer PIC151: Hook’s matrix [N/mm$^2$]

\[
\begin{bmatrix}
1.15 \times 10^5 & 7.06 \times 10^4 & 7.23 \times 10^4 \\
1.15 \times 10^5 & 7.23 \times 10^4 & 1.09 \times 10^5 \\
& & \\
& & \end{bmatrix}
\]

Symmetric

Piezoelectric constants: $e_{31} = e_{32} = 9.6 \times 10^{-6}$ C/mm$^2$
Dielectric constant: $d_{33} = 1.71 \times 10^{-8}$ F/m
Density $\rho = 7.80$ g/cm$^3$
The third example: Finally, a plate type of structure simply supported over all four edges is considered. The geometry of the model is given in Figure 8. A pair of collocated piezoelectric actuators (PIC 151) of thickness 0.1 mm is attached to a plate, the thickness of which is 1.59 mm. The plate is made of steel ($Y = 2 \times 10^5 \text{ N/mm}^2$, $\nu = 0.3$). Thus, we have a structural material of high mechanical stiffness, and the geometrical thickness factor is much smaller in this case in comparison to the first two geometries. The whole structure undergoes deformation, but only 7.7% of the structure is covered by piezoelectric patches, which differs a lot from the first two cases. However, it should be emphasized that this case corresponds at most to the usual application of distributed piezoelectric patches as actuators and sensors.

Figure 9 gives the deflection of the centerline of the plate when both patches are exposed to the same voltage of 1000 V. The results are nearly congruent and no difference can be observed in the figure due to the negligibly small influence of the additional stiffening effect. The maximal deflection without the stiffening effect is 0.3250 mm, and the same result with the effect included is 0.32499 mm, which is the difference in the order of 0.001%, and which is completely negligible. This example is originally proposed by Lammering.
and Mesecke-Rischmann (2003), who have developed a shallow shell element and considered the difference in the results yielded by the constant and analytically deduced linear functions for the electric field. However, they have reported a difference of 7% in this case, which is, in the opinion of the authors of the present work, too high.

CONCLUSIONS

Piezoelectric patches are more and more frequently used as active elements on thin-walled structures in order to obtain controllable dynamic response. Many researchers use different approximation functions for the electric field across the thickness of the patches often without an argumentation for their choice. This was a motivation for publishing the present investigation that considers closely the actual distribution of electric potential and electric field across the thickness of active elements, taking into account the kinematical assumptions. It was demonstrated through an analytical deduction that, besides the dependency on the difference of the electric potentials, the electric field also depends on the strain field or, in further instance, on the displacement field. Consequently, a first-order 2D theory yields a linear distribution of the electric field (a polynomial of the first order), whereby the linear term is related to the curvature of deformation. This further implies an additional stiffening effect noticeable in bending dominated problems.

By the way, it could be even demonstrated that a higher-order 2D theory would yield a function for the electric field of the same order as used by the theory itself.

Four different parameters influencing the magnitude of the additional stiffness are identified, and a set of bending-dominated problems is considered to demonstrate their influence. In the first case, the configuration set up is chosen so as to yield extremely high values of all the parameters except the piezoelectric material factor. Nevertheless, only after replacing the piezoelectric polymer PDVF by the piezoelectric ceramic PIC151 was the influence significant. Thus, the effect can be neglected in any case of application of piezoelectric polymers due to the small values of piezoelectric coupling constants. On the other hand, the last example mostly corresponds to a typical application of piezoelectric patches. The influence of the additional stiffening effect is in this case completely negligible, confirming the intuitive answer about the significance of the effect already discussed in the article. Therefore, for most of the typical applications, the effect can be neglected, but one should be aware of the parameters influencing it in order to recognize the case when the effect could play a more important role. This conclusion is especially important for nonlinear analysis, in which the inclusion of the effect would require much higher numerical effort, but the accuracy would not improve significantly for the present class of commercially available piezoelectric ceramics.

Figure 9. Deflection of the centerline of the plate.
Finally, we are witnesses of the continuous improvement of piezoelectric materials aiming at higher values of piezoelectric coupling constants. Hence, in the not-too-far future, it might be necessary to take the effect into account in order to describe the electromechanical coupling with satisfying accuracy.

REFERENCES


