Simulation Methods for Guided Wave-Based Structural Health Monitoring: A Review

This paper reviews the state-of-the-art in numerical wave propagation analysis. The main focus in that regard is on guided wave-based structural health monitoring (SHM) applications. A brief introduction to SHM and SHM-related problems is given and various numerical methods are then discussed and assessed with respect to their capability of simulating guided wave propagation phenomena. A detailed evaluation of the following methods is compiled: (i) analytical methods, (ii) semi-analytical methods, (iii) the local interaction simulation approach (LISA), (iv) finite element methods (FEMs) and (v) miscellaneous methods such as mass spring lattice models (MSLMs), boundary element methods (BEMs) and fictitious domain methods. In the framework of the FEM, both time and frequency domain approaches are covered and the advantages of using high order shape functions are also examined.

1 Introduction

The academic interest in structural health monitoring (SHM) related problems has been steadily increasing in recent years. These research activities are directed at (i) improving the structural functionality, (ii) decreasing the costs
and (iii) prolonging the effective live span of lightweight designs by means of non-destructive inspections and damage detection techniques. The main motivation is to develop an automatic continuous monitoring systems. Such a system should provide ample information on the structural state of the examined area of a component. These data can then be evaluated and be used to increase the service life of the structure and to reduce the operating and maintenance costs (O & M costs).

Among others, SHM systems are required to reliably detect and localize damages and, if possible, to assess its nature. From that point further measures can be initiated. Their ultimate purpose is to use the measured data to predict the remaining lifetime of the structure. Robust and reliable SHM systems accordingly enable the design engineer to either create safer structures or to fully exploit the material strength. Accordingly, the O & M costs can be reduced and truly lightweight designs can be created, respectively.

In the context of hi-tech industries, such as the aeronautical and civil engineering industries, strict safety requirements have to be enforced. This leads to high costs as highly trained and highly qualified personnel is needed to ensure their compliance. Typically, airplanes, offshore wind energy plants or bridges are in the focus of such endeavors. Different kinds of SHM systems have been proposed for these purposes during the past decade [25, 26]. Among them, ultrasonic guided wave-based SHM systems are a very promising approach and have found numerous applications recently.

From the authors’ point of view, efficient numerical tools are of utmost importance in the context of guided wave propagation analysis in lightweight structures. In order to devise a robust and reliable SHM system first the physical mechanisms governing the wave propagation in such structures have to be understood in detail. This necessary knowledge can be acquired by employing advanced numerical simulation approaches to support experimental investigations. Effects of particular interest such as the influence of ambient conditions and prestressing can be investigated straightforwardly with the help of numerical models if physical experiments are too costly or bulky. The simulation results can then be used to design, prepare and develop experimental techniques. Additionally, simulation methods can be used to design and to qualify the SHM systems itself.

To date, there is a wide range of numerical methods suitable for wave propagation analysis - each with its own advantages and disadvantages. Many of these approaches have been specifically tailored to meet the needs of wave propagation analysis. The authors feel that a review of the current state-of-the-art in computational methods for wave propagation analysis with a special focus on SHM-related problems is called for. The objective of the present article is therefore to provide an exhaustive overview dealing with numerical methods being suitable for high frequency structural dynamics. Since different problems are of interest when investigating guided waves for SHM applications, the present paper will evaluate the performance of each method with respect to the given task.

In the next section, typical SHM-related problems are discussed. Thereby the challenging tasks that are encountered when dealing with guided wave-based monitoring are illustrated. These information are used to derive the requirements which are the basis for assessing the different modeling techniques with respect to their ability to resolve the physical behavior of ultrasonic guided waves. The third and fourth sections deal with analytical and semi-analytical methods, respectively. The fifth section features finite difference schemes, with a special focus on the local interaction simulation approach (LISA). Various finite element methods (FEMs) are discussed in the sixth section of the paper. Here, both time- and frequency-domain approaches are presented. Additionally, high order shape functions are also discussed. Miscellaneous algorithms, that are only sporadically used for wave propagation analysis or that are not well-established yet, are mentioned in the seventh section. The article ends with a conclusion and an outlook on novel methods that are being developed. In the last section a short summary of the content of the article is also provided.

2 Guided waves for structural health monitoring

Designing SHM systems requires the expertise in a variety of different scientific disciplines. A well-grounded knowledge in mechanical engineering, electrical engineering, as well as in computer science, mathematics and physics, is essential [270]. Furthermore, a deep understanding of different material classes, signal processing techniques, communication processes in a transducer network, transducer design, damage assessment methods, etc. needs to be gained in order create robust and reliable SHM systems.

As already mentioned in the previous section, one promising approach for SHM in thin-walled structures is based on guided ultrasonic waves also referred to as Lamb waves. These are a special type of elastic waves that propagate in solid plates or layers with free boundaries. Therefore, displacements occur in the direction of wave propagation and perpendicular to it [256]. Only waves propagating in homogeneous isotropic thin plane plates with parallel stress-free surfaces were originally called Lamb waves [137]. In that special case, they are decoupled from shear-horizontal or Love waves. For this kind of waves displacements only occur in the plane of the plate and are perpendicular to its propagation direction [154]. However, the term Lamb wave is also frequently used in the SHM community to indiscriminately refer to waves propagating in anisotropic composite plates. In this particular case, Lamb and Love modes are coupled [9]. In accordance with the literature, the term Lamb wave is used in a broader sense to refer to ultrasonic guided waves in general.

Guided waves exhibit two features making them especially interesting for SHM applications. These are their short wavelengths in the high frequency range and their only slight loss of amplitude magnitude in relation to the traveled distance [241]. This results in a high sensitivity towards small perturbations of the waveguide such as damages and provides a possibility to scan bigger areas of the structure. ...
Additionally, two basic types of Lamb wave modes can be distinguished. Fig. 1a and Fig. 1b illustrate the displacement field corresponding to the fundamental symmetric mode \( S_0 \) and the fundamental antisymmetric mode \( A_0 \), respectively. These modes occur for every excitation frequency. When the value of the frequency \( f \) exceeds the value of the \( n \)-th cut off frequency high order symmetric modes \( S_1, S_2, \ldots, S_n \) and antisymmetric modes \( A_1, A_2, \ldots, A_n \) become propagating modes and significantly contribute to the received signal. These modes possess different wavelengths and propagation velocities. They accordingly interact with small perturbations differently and might convert into each other in the presence of damages or any other deviation of the plate geometry \([7, 9, 34]\).

![Fig. 1. Lamb wave mode shapes.](http://appliedmechanicsreviews.asmedigitalcollection.asme.org/)

In spite of their promising features, using Lamb waves for SHM applications has also some intrinsic difficulties. The propagating modes are dispersive in nature and can convert into each other in the presence of damages and other changes in the mechanical impedance. Due to the high propagation velocities also reflections from the structural boundaries may influence the wave behavior \([9, 241]\). Furthermore, their high sensitivity towards structural perturbations introduces random noise in the sensors related to imperfections in the bonding between actuators, sensor and the structure. Noise originating from environmental sources, local changes in temperature, inhomogeneities or material anisotropy also contributes to the received signals making them rather complex and hard to decode.

Motivated by the issues discussed above the following section provides a brief overview of the most important SHM-related research areas.

### 2.1 Lamb wave dispersion

The dispersive behavior of propagating waves is reflected in its spectrum, i.e. the relation between the wave vector and the frequency, which is also called dispersion relation. One way to explicitly describe the dispersive character of Lamb waves is to compute the solution of the dispersion equation for each frequency, in order to obtain the corresponding wave vectors/numbers. The result of this analysis can be illustrated with the help of dispersion curves, which are basically curves of the dependence between the wave vector or the phase/group velocity and the frequency. These are of great importance for the design of any Lamb wave-based SHM system, because through their study it is possible to answer questions such as, for which values of the frequency \( f \) and thickness of the plate \( h \) the impact of dispersion is negligible or not and which is the number of propagating modes in order to simplify the signal processing stage. For Lamb waves in plane, isotropic plates the dispersion relation is implicitly established by the Rayleigh-Lamb equation \([76]\).

The first comprehensive solution of the dispersion equation for each Lamb wave mode was provided by Mindlin in 1960 \([172]\). Using a digital computer, Gazis \([74]\) provided approximated solutions to the dispersion equation corresponding to propagating modes in 1958, which were computed more accurately by Viktorov in 1967 \([256]\). Viktorov also analyzed the problem of forced motion in the two-dimensional case. All these previous studies assumed plane strain conditions leading to a two-dimensional formulation of the problem. Moreover, the cited investigations were limited to isotropic materials. In Fig. 2 the phase and the group velocity dispersion curves for an isotropic aluminum plate are shown. The relation between the phase velocity \( c_p \), cf. Fig. 2a, or the group velocity \( c_g \), cf. Fig. 2b, and the value of the product of frequency and plate thickness \( f 
\cdot h \) is visualized for the first four Lamb wave modes \( S_0, S_1, A_0 \) and \( A_1 \). It can be proved that as the value of the product \( f 
\cdot h \) increases, the phase velocity asymptotically approaches the wave propagation velocity of Rayleigh or surface waves \([257]\). That is to say, as the excitation frequency of the wave or the plate thickness increases the ratio of wavelength to plate thickness decreases and as a consequence low order modes become less dispersive and behave like surface waves.

Dispersion curves for anisotropic materials can be obtained by solving the dispersion equation given in an analytical form, cf. Section 3. In plates consisting of layered composites, this problem is more complex and can be solved using the global transfer matrix method, the local transfer matrix method or semi-analytical approaches \([9]\), cf. Section 4. An alternative is to obtain the dispersion curves without the use of computational methods by means of experiments \([266]\).

### 2.2 Wave propagation in inhomogeneous anisotropic composite materials

Hitherto the majority of research activities involved plates made of isotropic materials \([103, 142–144, 176]\). Recent developments of SHM systems - especially for aeronautic structures - increasingly focus on multilayer composite materials. The behavior of ultrasonic guided waves is significantly more complex in such laminates \([249, 275]\). Inhomogeneity and material anisotropy add further difficulties to the process of damage detection. Due to these properties the wave behavior is also depending on the propagation direction \([63, 176, 232]\).

Another important research area is the field of material damping, especially when composite materials are consid-
2.3 Influence of environmental conditions

The influence of ambient conditions on the propagation of elastic guided waves is an active field of research [129, 140, 235]. Pre-stressing, temperature variations [40, 53, 160] and different moisture contents are among the environmental conditions that influence the propagation of ultrasonic waves. The last point is particularly critical for materials used in the offshore wind energy and aeronautical industries. Recently, several researchers studied the effect of the ambient temperature on the propagation of elastic guided waves [36, 40, 160, 175, 222]. Clarke [36] employed a baseline subtraction approach to account for temperature variations. His aim was to develop temperature-stabilized piezoelectric sensors for a mono-modal excitation of Lamb wave modes. Lu and Michaels [160] also proposed an approach to detect structural damages in the presence of unmeasured temperature changes. They showed that the primary effect of temperature on the recorded signals is a dilatation or compression of the time signal caused by the change in velocity and the thermal expansion of the structure under observation. A distortion of the wave form as a secondary effect was also observed. Andrews et al. [13] also investigated the influence of temperature on the group velocity and came to similar conclusions as reported above.

2.4 Damage detection

The research area of damage detection is fairly wide and involves intensive studies. Staszewski et al. [163, 239] conducted several studies concentrating on isotropic plates made of steel and aluminum. Both experimental and numerical results have been published. Miller et al. monitored the effects of corrosion in reinforced concrete [170]. Pruell et al. investigated the detection of plasticity and fatigue damage with Lamb waves [200, 201]. In typical lightweight designs made of multi-layer composite materials non visible damages occur quite often during the service life. SHM systems should therefore be able to detect different types of damage, such as impact damage [178] or delaminations/debondings [209, 245, 252] which are not encountered in metals. By way of an example, Liu et al. [152] conducted simulations of ellipsoidal crack with Chebyshev pseudospectral Mindlin plate elements.

2.5 Piezoelectric transducers

Piezoelectric materials constitute another important research topic with respect to SHM applications. Piezoelectric transducers are often used to excite and receive ultrasonic guided waves due to their low costs and an easy integration into existing structures. This research area offers a wide variety of contributions. Important contributions are listed in the following paragraphs.

Giurgiutiu et al. [76] conducted investigations into different types of thin piezoceramic transducers. Ostachowicz et al. [9, 190, 225] studied the actuator design or phased array morphologies, whereas Moulin et al. applied embedded
piezoceramics rather than surface-bonded ones [179]. Since piezoelectric transducers are often glued to the structure the influence of the adhesive layer between the actuator and the plate is of interest as well. Sirohi et al. [27, 203, 204, 234] focused on the so-called "shear lag" effect. This effect describes a reduction in the transfer of the shear stresses from the actuator to the host structure caused by the adhesive layer. Ha et al. [89, 199, 270] found that the dynamics of the piezoelectric actuator strongly depends on its thickness and therefore it significantly influences the amplitudes of the excited Lamb waves. Blackshire et al. [138, 197, 198] studied the influence of debonding. Moreover, Huang et al. showed that asymmetric eigenforms of the piezoceramic transducer cause an asymmetric wave field amplitude characteristic [104].

2.6 Signal analysis

Due to the high complexity of measured and simulated wave signals, the data needs to undergo further processing. That is why signal processing is a very important, if not the most important, step in online monitoring approaches. To compensate for dispersive effects Giurgiuțiu [75] proposed mode tuning. Wilcox [269] presents a technique "that enables dispersive time-domain signals to be mapped to propagation distance while removing the effect of dispersion." Michaels and Michaels [169] proposed damage detection in aluminum plates using an image fusion technique and Ng and Veidt [184], among several others, carried out damage localization in composite laminates. They used a network of sensors to sequentially scan the structure and to reconstruct a damage image. These few examples highlight the wide field of signal processing for Lamb wave-based SHM. Since a complete review of signal processing techniques is not in the scope of the current article the interested reader is referred to [207, 236] and the references cited therein.

2.7 Simulation methods

This brief overview highlighted the complexity and versatility of research problems encountered when dealing with the development of online monitoring systems based on ultrasonic guided wave propagation. From this we can infer potential research areas in the context of wave-based SHM applications:

1. Dispersion analysis,
2. Modeling of complex damage scenarios,
3. Interaction of Lamb waves with obstacles,
4. Actuator-sensor-models (including bonding conditions),
5. Modeling of Lamb waves in heterogeneous materials with a complex microstructure,
6. Environmental effects on the wave propagation,
7. Optimization of transducer networks,
8. Propagation of higher order wave modes.

Regarding these research fields it can be concluded that not only one numerical method is able to efficiently solve each single problem. Therefore, a wide variety of established numerical methods for wave propagation analysis will be discussed and their performance with respect to SHM-related problems will be evaluated. Appropriate criteria for the assessment are developed in the following paragraphs. For more detailed information about SHM, the reader is referred to the works by Su et al. [241, 242], Raghavan and Cesnik [207] and the Encyclopedia of structural health monitoring edited by Boller et al. [25]. However, in these references the broad variety of numerical methods being suitable for wave propagation analysis are only briefly covered. The current paper, therefore, bridges the gap between experimental studies and numerical analysis in the framework of wave-based continuous monitoring approaches.

There is a wide variety of computational techniques employed to solve different types of problems arising when dealing with SHM applications. Yet, it is difficult to provide a general statement that can answer the question concerning the most powerful numerical method. In this context the terms suitable and efficient are only used according to the problem under consideration. There are general approaches that are, in principle, able to solve all problems discussed in the course of the previous paragraphs. However, they are not necessarily efficient or recommended in each case. In the course of the present paper the most important aspects of different widely-used and accepted computational methods are summarized and it is pointed out when and why to choose one among the others. The criteria taken into account for the assessment are the following:

1. Capability to describe the Lamb wave dispersion behavior: This basically reduces to the possibility to efficiently solve the dispersion equation. The solution of the dispersion equations - visualized by dispersion curves - allows to describe the distortion of signals due to the physical dispersion and to determine the values of the cut-off frequencies. This is very useful for SHM applications dealing with higher order Lamb modes. In this case, methods formulated in the frequency domain like Analytical and Semi-analytical methods are the most reliable.

2. Possibility to describe Lamb wave propagation in complex, heterogeneous and/or anisotropic materials: This is of great importance for applications dealing with ultrasonic wave propagation in anisotropic composite plates. The possibility to implement different constitutive material models and the influence of the ambient conditions in the numerical simulation contributes to a more reliable and accurate analysis. Methods of different kind can be recommended to tackle these problems depending on other specifications of the problem such as the geometry and boundary conditions for example.

3. Flexibility to model arbitrary geometries: The geometry of the waveguide significantly affects the wave propagation behavior in general. The shape of defects,
sensor and actuators also influences the traveling signals. Finite element-based methods are commonly recommended in these situations due to their ability to accurately describe the geometry of the structure using body-fitted meshes.

4. Calculation time and memory storage requirements: The calculation time and memory storage requirements to obtain a certain degree of accuracy are always an important issue when working with numerical simulation methods. When several methods can be used for the same task it is always preferable to choose the one with the least requirements provided that a similar accuracy can be achieved. Analytical, semi-analytical and parallelizable methods are favored with respect to these criteria.

A wide variety of papers is listed alphabetically in the reference section. Nonetheless, this survey can only be incomplete and the authors wish to apologize, in advance, for any inadvertent omission of relevant publications. We have endeavored to mention all important contributions to the field of numerical wave propagation analysis in the context of SHM-related problems and hope we have managed to provide the reader with a useful survey.

3 Analytical methods

The beginnings of the theoretical studies concerning Lamb waves date back to 1889 with Lord Rayleigh, who first modeled the propagation of elastic waves along a guided surface of an isotropic homogeneous semi-infinite solid [211]. These waves are known today as surface waves or Rayleigh waves. In 1911, Love added a second surface that is parallel to the first one in order to simulate horizontally polarized waves, i.e. shear horizontal waves [154]. Based on the work of Lord Rayleigh, Horace Lamb published his seminal work On Waves in an Elastic Plate in 1917 [37]. Lamb studied the propagation of guided waves between two parallel and plane surfaces with the polarization of the displacements parallel and perpendicular to the direction of propagation. After his work was published, elastic waves in plates were named after him. Like Rayleigh, Lamb only considered the displacement field in the sagittal plane where conditions of plane strain were assumed. In the aforementioned works, only a single frequency was analyzed in order to provide the dispersion relations between wave number and circular frequency. The displacement field under external loads was not described and, although the multi-modal behavior of Lamb waves was only implied and not studied in depth, Lamb established the theoretical principles to model such waves. The main motivation behind his pioneering work was related to seismic problems. Lamb’s problem was not taken up again until 1945, when Osborne and Hart investigated Lamb waves excited by underwater explosions [187]. At this point, it became imperative to solve the Rayleigh-Lamb dispersion equation in order to arrive at the final solution of the displacement field. Holden [98] and Mindlin [171] employed upper and lower bounds together with asymptotic methods in 1951. Onee [186] extended this work in 1955 to derive the qualitative behavior of the real branches of the Lamb wave dispersion curves, corresponding to propagating Lamb modes. The computation of bounds and the qualitative behavior of imaginary branches, corresponding to evanescent Lamb modes, was first performed by Lyon [162] in 1955. Mindlin and Medick [173] established the existence of complex branches and their corresponding phase velocities for real frequencies in 1959. In 1958, Gazis [74] used a digital computer to give approximated solutions to the dispersion equation corresponding to propagating modes. Later on, in 1960, Mindlin came up with a comprehensive solution for the dispersion equation for each Lamb wave mode [172]. These were computed very accurately by Viktorov in 1967 [256]. Viktorov also analyzed the problem of forced motion in the two-dimensional case. In all these previous studies plane strain conditions were assumed, leading to a two-dimensional formulation of the problem. In 1973, Achenbach presented a study on the formulation for the problem of forced motion and the response of a plate under vertical point forces applied on the surface [4]. Graff realized an extension to the three-dimensional problem by studying circular crested waves in 1975 [83]. Once Lamb waves found an application in non-destructive analysis applications during the last decades, efforts to model waves in plates have multiplied. Achenbach [1–3, 5, 6] used the concept of the membrane carrier wave together with mechanical reciprocity to describe the displacement field in isotropic plates.

In order to extend these results to general three-dimensional plates and different load distributions the Fourier transform and the Cauchy’s theorem of residues can be used. For this approach it is referred to the works of Gomilko et al. [79], Shi [233], Raghavan et al. [205, 206], Giurgiutiu [75, 76], von Ende et al. [262] and von Ende and Lammering [259, 260]. In the work of Wilcox [268], the excitation matrices are defined to model Lamb waves excited by line and point forces. Modal analysis has also been taken into account in order to provide analytical solutions, as in the work of Jin [112]. For these formulations in the frequency domain, it is essential to solve the dispersion equation that relates the wave-vectors and the frequency. Higher order plate theories have also been used in order to obtain a good approximation of the dispersion equation, which can be solved analytically as in the work of Yang and Yuan [276]. Another approach is to solve the exact dispersion equation numerically and obtain an accurate approximation of the wave number for each frequency, as done in [257].

Wang and Achenbach [264] dealt with the application of these techniques to the case of homogeneous materials of general anisotropy in the wave-vector frequency domain. There are several contributions on the topic of elastic wave propagation in layered elastic plates of general anisotropy. The works of Nayfeh [182, 183], in which dispersion relations are provided for generally anisotropic layered composites, represent an important breakthrough in this context. The dispersion relation between the wave-vector and the fre-
frequency is determined using the local matrix method [158]. During the last few years it has been replaced by the global matrix method. This approach solves the problem of the numerical instability encountered when employing the local matrix method for high values of the product thickness multiplied by the frequency [265]. Recent analytic approaches are based on the use of the Green’s tensor for point forces that are applied on the surface of the three-dimensional model of the plate. This Ansatz can be seen in the works of Karmazin et al. [115, 116] and Glushkov et al. [77, 78]. Some hybrid approaches have also been formulated as in the work of Velichko and Wilcox [255] where the Green’s function for a laminated plate is obtained using modal analysis.

These mathematical methods form the basis for several studies of theoretical and experimental features of wave propagation in composite plates, such as the investigation of the dispersive behavior of guided waves [266] and the research concerning the excitation and focusing of Lamb waves [30], among others.

### 3.1 Analytical methods in the frequency domain

#### 3.1.1 Outline of the problem

The formulation of the mathematical problem is based on the assumption of a plane composite plate of thickness \( h \). The mid-plane of the plate coincides with the \( x_1-x_2 \) coordinate plane of a three-dimensional Cartesian coordinate system and the \( x_3 \)-axis is perpendicular to it. The upper and lower surfaces of the plate are located on the planes \( x_3 = -h/2 \) and \( x_3 = h/2 \), respectively. The plate domain is denoted by \( \Omega \) which corresponds to all points \((x_1,x_2,x_3)\) of \( \mathbb{R}^3 \) so that \(-\infty \leq x_1,x_2 \leq \infty \) and \(-h/2 \leq x_3 \leq h/2 \). It is also assumed that no body forces exist and that the loads are only applied on one surface of the plate. Considering these conditions, the starting point for the formulation of analytical methods are the equilibrium equations in the frequency domain and the corresponding boundary conditions

\[
c_{ijkl} u^i_{kl} + \rho \omega^2 u^i = 0, \tag{1}
\]

\[
c_{ijkl} u^i_{kl} |_{x_3=-h/2} = T^i, \tag{2}
\]

\[
c_{ijkl} u^i_{kl} |_{x_3=h/2} = 0, \tag{3}
\]

where \( c_{ijkl} \) are the components of the elasticity tensor and \( \rho \) is the mass density. These material properties may depend on the position vector \( \mathbf{x} \). The quantities \( u^i_r \) and \( T^i_r \) marked with an asterisk to denote the time Fourier transforms of the components of the displacements \( u_k \) and the loads \( T_r \), respectively. The angular frequency is denoted by \( \omega \). It is assumed that \( u^i_r \) and \( c_{ijkl} u^i_{kl} \) are continuous functions of the position vector \( \mathbf{x} \). A subscript \( j \) after a comma represents the partial derivative with respect to \( x_j \) and the rule of summation over repeated subscripts (Einstein convention) is employed.

In general, the modeling of elastic waves is not restricted to plates with a plane geometry, as presented at this point. Plates with curved geometry have been considered in [52, 87, 210, 251], where the propagation in pipes is investigated. Wave propagation in more generally curved geometries is dealt with in the works of Gridin and Craster [87] and Harris [93]. In [257] cases involving body forces are studied. In the same manner, different boundary conditions are employed, cf. Eqs. (2) and (3). The works of Lowe et al. [159] and Nayfeh [183] investigate wave propagation in plates with boundary conditions corresponding to plates immersed in fluids (water or air). In [183] the boundary conditions that model the common interface of the plate and an elastic solid are investigated. Barshinger and Rose [17] studied the wave propagation in an elastic hollow cylinder coated with a viscoelastic material. In [257, 258] Vivar-Perez et al. also included the boundary conditions for modeling a piezoelectric patch bonded to the surface of the plate. Boundary conditions where one surface, or part of it, is fixed are dealt with in [115]. In this paper, the model (cf. Eqs. (1)-(3)) has been used to illustrate the standard procedure common to most of the described cases.

Using this procedure the solution of the boundary value problem (Eqs. (1)-(3)) can be written in terms of a convolution integral:

\[
u^i_r(\mathbf{x},x_3) = \int_{\mathbb{R}^2} G^i_{lm}(\mathbf{x} - \mathbf{\bar{y}},x_3) T^l_m(\mathbf{\bar{y}}) \, d\mathbf{\bar{y}} \tag{4}\]

where \( G^i_{lm} \) is the Green’s matrix, response tensor or fundamental solution. The upper bar in \( \mathbf{\bar{y}} \) and \( \mathbf{\bar{y}} \) denotes that these vectors vary only in the \( x_1-x_2 \) plane. For most of the applications employing analytical methods, the main goal is to compute the response tensor \( G^i_{lm} \). In cases where point forces are applied to the surface of the plate, Eq. (4) gives an analytical formula for \( u^i_r \) [75, 76]. If it is dealt with a distribution of forces in the surface then the integral in Eq. (4) is calculated using quadrature formulae [104, 257].

#### 3.1.2 Response tensor and wave-vector domain

In order to find the fundamental solution the Fourier transform is applied. The transformation is performed from the space domain, denoted by the position vector \( \mathbf{x} \), to the wave-vector domain, denoted by the Fourier variable or wave-vector \( \vec{\xi} \)

\[
\hat{G}(\mathbf{x}, \vec{\xi}) = \int_{\mathbb{R}^2} G^*(\mathbf{x},x_3) e^{-i\vec{\xi} \cdot \mathbf{\bar{x}}} \, d\mathbf{\bar{x}}. \tag{5}\]

The length of the vector \( \vec{\xi} \) is denoted by \( \xi \) and the angle between \( \vec{\xi} \) and the \( x_1 \)-axis is denoted as \( \phi \). The components of the tensor \( \hat{G} \) satisfy the following system of ordinary differential equations in the variable \( x_3 \)

\[
c_{ijkl} \hat{G}_{3m,33} + i\xi^2 (\hat{c}_{ijkl} + \hat{c}_{iklj}) \hat{G}_{3m,3} - (\xi^2 \hat{c}_{ikl1} - \omega^2 \rho \delta_{ik}) \hat{G}_{3m,3} = 0 \tag{6}\]
where the coordinate transformation matrix $\alpha$ is given by

$$
\alpha = \begin{pmatrix}
\cos \varphi & \sin \varphi & 0 \\
-\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 1
\end{pmatrix}.
$$

The system of equations Eq. (6) is constrained to the boundary conditions given by Eqs. (7) and (8). It can be solved analytically, leading to an expression of the response matrix $\hat{G}$ in the wave-vector domain

$$
\hat{G} = \frac{g(x_3; \bar{\xi}, \omega)}{\det D(\bar{\xi}, \omega)}.
$$

For the specific case presented here, including the boundary conditions given in Eqs. (7) and (8), the components of the matrix $D$ have the following form

$$
D_{ij}(j) = \begin{cases}
\bar{\xi} (\kappa(j) \bar{\xi} + \bar{\xi} j_{13}) \\
\times \bar{g}_k(j) \exp(ikj(j)\xi d/2) & \text{if } i = 1, 2, 3 \\
\bar{\xi} (\kappa(j) \bar{\xi} - j_{13}) + \bar{\xi} j_{13} \\
\times \bar{g}_k(j) \exp(-ikj(j)\xi d/2) & \text{if } i = 4, 5, 6.
\end{cases}
$$

where indices $i$ and $j$ are written in brackets to denote that they vary from 1 to 6. The values of $\kappa(j)$ and $\bar{g}_k(j)$ for $j = 1, 2, \ldots, 6$ and $k = 1, 2, 3$ in Eq. (12) are determined by a quadratic eigenvalue problem

$$
[\kappa^2 \bar{\xi} + \kappa(\bar{\xi} \bar{\xi} + \bar{\xi} j_{13}) + \bar{\xi} j_{13} - c^2 \bar{\xi}] \bar{g}_k = 0,
$$

where $c = \omega/\bar{\xi}$. The components of the matrix $g$ in Eq. (11) can be written in terms of $D$, $\kappa(j)$ and $\bar{g}_k(j)$

$$
g_{km} = \sum_{j=1}^{6} \bar{D}_{ij}(j) \bar{g}_k(j) \exp(ikj(j)\xi x_3), \quad k, m = 1, 2, 3
$$

where $D_{ij}(j)$ are the components of the adjoint matrix of $D$ i.e. $D_{ij}(j)$ is the cofactor of $D$ corresponding to the $m$-th row and the $j$-th column for $m = 1, 2, 3$ and $j = 1, 2, \ldots, 6$.

Eq. (11) is also valid for problems modeling laminated composites of general anisotropy and other types of boundary conditions. In these cases, it is necessary to contemplate the bonding conditions between layers and additional boundary conditions. In order to obtain $D$, $\kappa(j)$ and $\bar{g}_k(j)$ the boundary conditions in Eqs. (7) and (8) have to be applied and subsequently Eq. (11) is evaluated [115, 183]. There are several methods reported in the literature to solve this problem. The most common ones are the transfer matrix method and the global matrix method [158]. A main drawback of the transfer matrix method, originally proposed in 1950 [248], lies in the numerical instability of the solution for large values of $f$·$h$, also known as the "large $f$·$h$ problem" [64]. The problem of instability is solved by the global matrix method, originally formulated in [121]. However, a costly assembly process of a large matrix has to be executed. This method is very reliable [115] - albeit with the disadvantage that it is slower than the local transfer matrix. Moreover, when there are a lot of component layers in the composite the global matrix method results in a very large system matrix [158]. Wang and Rokhlin proposed another way of circumventing the problem of the intrinsic instabilities of the standard transfer matrix method in [265]. They use a stable reformulation of the local matrix method and make use of an iterative procedure to obtain the global stiffness matrix from the stiffness matrices of each separate layer. This recursive algorithm has proved to be robust, and it is slightly more efficient in terms of computational costs than the transfer matrix method.

In order to be able to employ Eq. (11), first the values of $\omega$ and $\bar{\xi}$ have to be determined. The values of $\bar{\xi}$ which satisfy the following equation

$$
\det D(\bar{\xi}, \omega) = 0
$$

are singular values of Eq. (11) i.e. they are poles of the function $\bar{G}(x_3; \bar{\xi}, \omega)$ for each value of $\omega$. Eq. (15) is known as the dispersion equation of the medium, and it is usually solved in such way that $\bar{\xi}$ is found in terms of $\varphi$ and $\omega$

$$
\bar{\xi}_n = \xi_n(\varphi, \omega).
$$

The numbers $\xi_n$ are the roots of Eq. (15) and are in general infinite. For the boundary conditions considered here and for each fixed value of $\varphi$ and $\omega$ there are a finite number of real roots corresponding to the propagating modes, as well as an infinite number of imaginary roots corresponding to the evanescent modes. Curves showing this dependence - Eq. (16) - are called spectrum or wave number dispersion curves, and they are essential when investigating of the propagation of Lamb waves in multilayered media [30, 116, 266]. Once $\bar{G}$ has been calculated, it is possible to determine the response tensor $G^*$ using the inverse Fourier transform.
over the interval real axis and computes the remaining integral numerically. The technique developed in [135] removes the poles from the path of poles of the function \( \hat{G} \). In this way, it is possible to use this method, first the dispersion equation, cf. Eq. (15), to evaluate the inverse Fourier integral in Eq. (17). The expression for the integral with respect to \( \xi \) in Eq. (17) is replaced by the integral

\[
\mathbf{u}(\bar{x}, x_3, t) = \frac{e^{i\eta}}{2\pi} \int_{-\infty}^{\infty} \mathbf{u}(\bar{x}, x_3; \omega) e^{i\omega t} d\omega.
\]  

(18)

Using this method avoids the complications of finding the poles of the function \( \hat{G} \) as well as also the singularities present when transforming the response from the frequency domain to the time domain [149].

The third kind of method, which can also be combined with the previous ones, consists of evaluating the integral along the real axis by choosing a complex path that bypasses the poles of the integrand laying on the real axis [149]. The complex path should be chosen in such a way that the poles are kept out of the integration domain. Although it is advisable to have some idea of the location of the poles, it is not necessary to calculate their exact position on the real axis in advance.

The integration with respect to \( \varphi \) in Eq. (17) is less problematic once the integration with respect to \( \xi \) has been completed. The most popular techniques used for solving this kind of integral are asymptotic methods like the stationary phase method [30, 255] and the usual numerical quadrature formulas, as proposed in [32]. The first method is more convenient for cases where the results are needed at an observation point \( \bar{x} \) that is further away from the points \( X \) where the external forces are applied and only the propagating modes have to be taken into account. It is also possible to execute a direct integration in the case of isotropic homogeneous or laminated plates [14, 75, 76, 257, 260].

An alternative to compute the double integral in polar coordinates, cf. Eq. (17), is given in [16]. Although the Fourier transform is the most commonly used transformation, other analytical integral transform methods, such as the Laplace transform [84], are also feasible in order to directly obtain the response of the plate in the time domain.

3.1.3 Inverse Fourier transform and contour integral

One of the main difficulties in analytical methods that are employed to model the propagation of Lamb waves is the evaluation of the inverse Fourier integral in Eq. (17). The accurate evaluation of this integral is very important, since it provides the response matrix of displacements \( \hat{G} \) in the spatial domain for each frequency from the analytical closed form expressions of \( \hat{G} \) in Eq. (11). The two main issues, arising during the evaluation of this integral, are the complexity of the expression of the integrand \( \hat{G} \) and the existence of poles of the function \( \hat{G} \) that are located in the integration path \( \xi \in [0, +\infty] \).

Several methods have been developed to solve the problem presented by these complexities. The first relies on the use of a path or contour in the complex \( \xi \)-plane which approaches the integration axis \( \xi \in [0, +\infty] \) in the limit case. In this approach, the integration contour employed is usually closed, contains the integration axis \( \xi \in [0, +\infty] \) and bypasses the poles in the integration axis according to the principle of minimal absorption [30, 115, 116]. Once this integration contour has been determined, the limit case is analyzed by applying Cauchy’s theorem of residues [14, 75, 76, 257, 260]. This method is highly advantageous, since it provides closed form expressions for the integral with respect to \( \xi \) in Eq. (17). To utilize this method, first the dispersion equation, cf. Eq. (15), has to be solved in order to find the dependence given in Eq. (16). It must be noted, that for some cases, this might involve a great deal of computational effort in comparison to other methods commonly used to solve this problem [115]. The technique developed in [135] removes the poles from the real axis and computes the remaining integral numerically over the interval \( \xi \in [0, +\infty] \). For this method it is imperative to determine the location of the poles on the real axis. Therefore, it is necessary to find them before one is able to apply the methodology described in [135].

Another technique, that avoids the need to calculate the poles of the function \( \hat{G} \) in advance, involves the introduction of a small material dissipation such in a way that the poles are shifted away from the real \( \xi \)-axis. The problem that remains is to calculate the integral without knowing the poles’ location. Commonly this computation is performed numerically with the help of adaptive quadrature formulas [31, 32, 149, 218, 274]. This method is closely related to the use of complex frequencies to obtain the response of the plate in the time domain using the exponential window method. The exponential window method is basically a numerical-analytical approach designed to avoid the difficulties connected with the singular integrals of the functions with poles on the real axis. According to this method, the inverse Fourier transform

\[
\mathbf{G}^*(\bar{x}, x_3; \omega) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \frac{g(x_3; \xi, \omega) e^{i\xi \cdot \bar{x}}}{\det \mathbf{D}(\xi, \omega)} d\xi
\]

(17)

\[
= \frac{1}{4\pi^2} \int_{0}^{2\pi} \int_{0}^{\infty} \frac{g(x_3; \xi, \varphi, \omega) e^{i\xi \cdot \bar{x}} \cos \varphi}{\det \mathbf{D}(\xi, \varphi, \omega)} \xi d\xi d\varphi.
\]
1. the qualitative behavior of Lamb wave propagation can be described merely by studying the analytical expressions,
2. the modeling of plates with infinite dimensions is straightforward,
3. the influence of each Lamb wave mode can be simulated and analyzed separately,
4. and the effects of a frequency dependent damping are easy to integrate into the formulation.

Analytical methods are highly accurate, but they have the inherent disadvantage of having been developed for specific geometries of the plate and/or for specific distributions of loads only.

There have been a number of attempts to describe the propagation of elastic guided waves in other geometries apart from plane plates using analytical methods. These geometries include weakly curved plates [87,93], cylindrical curved plates [251] and pipes [17,210], for example. However, analytical methods are still unable to model arbitrary complex geometries, which is one of their biggest limitations.

Therefore, the development of numerical simulation tools using analytical methods moves towards hybrid formulations. These methods deploy discrete approximation approaches in regions of the structure where analytical methods cannot be used and couple them with an analytical method that describes the wave field in the rest of the plate [257,258]. Other extensions of these methods use a combined formulation of analytical methods and FEMs throughout the entire geometry of the structure. Here, in order to model complex geometries, the structure is not divided into regions according to whether analytical methods can be applied or not [81,279].

4 Semi-analytical methods

The semi-analytical finite element method (SAFE) is a suitable candidate for dealing with the variation of the material properties in the thickness direction of a plate while retaining advantages known from purely analytical approaches, such as the infinite dimensions of the structure. With this method, the material variation along the plate thickness direction is described using a finite element (FE) approach while, in the wave propagation direction analytical, complex-valued exponential functions are used. This method exploits the benefits of numerical and analytical approaches. For plane waveguides, only one-dimensional (1D) elements are needed. Each material layer in the plate is represented by at least one 1D element. Two-dimensional (2D) elements, however, are needed for modeling complex, three-dimensional waveguides such as rails and rods [73,95,155,167,253].

4.1 Formulation of the semi-analytical finite element method

The formulation of the SAFE method is very similar to its companion the FEM. The main difference lies in the interpolation function used for the elements. Consider an infinite plate which is discretized along its thickness direction using 1D elements, as shown in Fig. 3. The plate thickness is denoted by h. The x3-axis is taken to be perpendicular to the plate and guided waves propagate in the direction of the x1-axis. Displacements within the element \( u^{(e)} \) are approximated from their nodal displacements \( q^{(e)} \) as

\[
u^{(e)}(x_1,x_3,t) \approx N(x_3)q^{(e)} \cdot e^{i\omega t - ikx_1}
\]

where \( k \) and \( N(x_3) \) are the wavenumber and the displacement shape function, respectively. The element strains can be obtained from the displacements. Applying Hamilton’s principle and summing the contributions from every element, as in the standard FEM, the SAFE governing equations for a plate without external loads are obtained

\[
[k^2K_3 + kK_2 + K_1 + \omega^2 M]U = 0,
\]

where \( K_1, K_2 \) and \( K_3 \) are the global stiffness matrices and \( M \) is the global mass matrix. Details regarding these matrices are given in [19]. \( U \) is the vector of global displacements at the circular frequency \( \omega \). Eigenvalues of the system corresponding to Eq. (21) are the values of \( k \) which make the matrix \( k^2K_3 + kK_2 + K_1 + \omega^2 M \) singular. This dependence between the wave number \( k \) for each mode and the angular frequency \( \omega \) constitutes the dispersion equation. By solving the homogeneous system for each of these values \( k \), their corresponding eigenvectors \( \psi \) can be found. They provide the displacement field within the plate for each mode. Plots of wavenumber \( k \) versus frequency \( \omega \) produce the dispersion curves according to this method. The SAFE method has been mainly used to obtain the dispersion curves of isotropic and composite plates. More complex waveguides, such as rods, wires and rail road track have also been investigated [19,41,73,96,156,216]. The effect of initial axial loads on the dispersion curves is considered through additional terms in the formulation [157]. As in conventional FE applications, both a h- and/or p-refinement can be considered to increase the accuracy of the simulations. This is especially important when an increased accuracy is needed in order to simulate the displacement field of higher order modes.
that occur above a certain cut-off frequency [33]. Simulations to model the plate’s response to an external force and to a piezoelectric excitation have been done using the SAFE method in two-dimensional [41, 180] and three-dimensional (3D) cases [30, 255]. These simulations solve the wave propagation in infinite plates, since the SAFE method implicitly considers the plate to be infinite in its formulation. Additional effects on the wave propagations i.e. reflections and transmissions from damages, actuators or boundaries would make the wave propagation behavior more complex and must be added separately. The SAFE method can be combined with the FEM, the boundary element method (BEM), and other computational approaches to determine the reflection and transmission behavior of ultrasonic guided waves at geometrical perturbations of the infinite plate such as damages, transducers and boundaries [7, 8, 94, 117, 118, 150, 246, 250]. In these cases, the infinite plate section is modeled using SAFE, while the damages or boundaries are modeled using FEM or BEM, respectively, as illustrated in references [7, 8].

4.2 Variations of the semi-analytical finite element method

The SAFE method is also known as the stiffness method [42] and is closely related to both the strip element method [149] and the thin element method [263], which is widely applied to model seismic wave propagation. Another semi-analytical method is the wave finite element method (WFEM) [108, 130, 164, 165]. This method relies on the periodicity of the waveguide geometry, which is modeled using only a periodic section of it. The great benefit of this approach is that the periodic section can be modeled using available commercial FEM software. The stiffness and mass matrices of the periodic section are calculated with the FEM and used to obtain the relationship between wave number \( k \) and circular frequency \( \omega \), providing a way to determine the dispersion curves as well as the mode shapes. The output of the WFEM is the same as that of the SAFE method, therefore it is possible to extend the analysis carried out using SAFEs with the WFEM.

Consider a periodic section \( n \), as shown in Fig. 4. This periodic section is modeled using the FEM. Nodes at the periodic boundaries and inside the FE meshes are denoted by \( r \), \( i \) and \( l \) respectively. The dynamic stiffness matrix for the periodic section at each circular frequency \( \omega \) is

\[
D(\omega) = K - i \omega C - \omega^2 M,
\]

where \( K \), \( M \) and \( C \) are the stiffness, mass and damping matrices, respectively. The dynamic stiffness matrix relates displacements \( \mathbf{u} \) and forces \( \mathbf{f} \) in the periodic section as

\[
D\mathbf{u} = \mathbf{f}.
\]

According to Fig. 4, the vectors of displacements \( \mathbf{u}_r \) and \( \mathbf{u}_l \) and the vectors of forces \( \mathbf{f}_r \) and \( \mathbf{f}_l \) related to the degrees of freedom from the nodes at the boundaries \( r \) and \( l \), respectively. They are enforced to satisfy special conditions derived from the periodicity of the waveguide. Assuming a harmonic wave propagation in \( x_1 \)-direction and a complex wave number \( k \), the displacements and the forces are given as

\[
\mathbf{u}_r = \exp(k \Delta L) \cdot \mathbf{u}_l, \quad (24)
\]

\[
\mathbf{f}_r = \exp(k \Delta L) \cdot \mathbf{f}_l. \quad (25)
\]

This leads to an eigenvalue problem

\[
T(\omega) \begin{bmatrix} \mathbf{u}_n^{(r)}(\omega) \\ \mathbf{f}_i^{(r)}(\omega) \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{u}_n^{(l)}(\omega) \\ \mathbf{f}_i^{(l)}(\omega) \end{bmatrix} = \exp(k \Delta L) \begin{bmatrix} \mathbf{u}_n^{(r)}(\omega) \\ \mathbf{f}_i^{(r)}(\omega) \end{bmatrix}. \quad (26)
\]

Details of matrix \( T(\omega) \) are given in [130]. This eigenvalue problem has \( 2N \) solutions with \( N \) denoting the number of degrees of freedom. From the obtained eigenvalue \( \lambda \), the complex wave number is computed by

\[
k = \frac{\ln(\lambda)}{\Delta L}. \quad (27)
\]

The relation between the wavenumber \( k \) and the frequency \( f \) determines the dispersion curve. From that the displacement field can be computed.

4.3 Assessment of semi-analytical finite element methods

The SAFE method is a numerical method in the frequency domain and requires higher computational costs to achieve a prescribed level of accuracy than analytical methods. Compared to the FEM the computational requirements are nonetheless significantly lower. Similar to the analytical
methods an inverse Fourier transform must be numerically performed to recover the response of the plate in the time domain. Another main advantage shared by both analytical and semi-analytical methods is that each mode can be considered individually. This allows a detailed analysis of the wave propagation behavior, since specific modes of interest can be considered separately in further stages of the analysis [166].

The interaction of propagating waves with regions of the structure deviating from the plate-like geometry can be easily modeled. On the other hand, structural boundaries and damages must be discretized separately in order to include effects such as reflection, transmission and mode conversion. Therefore, it is only possible to achieve accurate simulation results for arbitrary geometries if the SAFE method is coupled with another numerical approach. Performing a 3D force response analysis requires certain approximations to be made, as illustrated in [30, 181, 255], which results in accurate simulations at low values of \( f \cdot h \) and in isotropic plates only. The distance between the source of the excitation and measurement points has no effect on the calculation effort in the analytical and semi-analytical method. In order to capture all possible wave propagation features the standard FEM is still a more suitable approach compared to the SAFE method, if the waveguides possess complex geometries. Semi-analytical or analytical methods are an adequate choice

1. to investigate the behavior of specific modes,
2. to efficiently provide a plot of dispersion curves,
3. to calculate the simulation results in small areas of interest such as the regions where sensors are placed.

provided that the geometry can be accurately described. Generally speaking, the SAFE method is more effective and efficient than the standard FEM in terms of the calculation time and the computer memory storage to achieve the same degree of accuracy.

5 Finite difference schemes

The finite difference method (FDM) is generally based on Taylor expansions and its direct substitution into the governing equations. This numerical approach needs a grid of points to be set up in a structured way. According to this method the Taylor expansion of a continuous function around some point \( x \) is truncated after an arbitrary number of terms what determines the order of accuracy of the finite difference formula. Depending on the dominant power in the truncation error term, finite difference formulas are said to be of first, second or third order and so on. Using this procedure it is possible to derive finite difference formulae of any order for every derivative. It only depends on the number of mesh points being used. An increased number of points also increases the accuracy of the formula. Backward, forward, central or mixed finite difference schemes can be distinguished depending on the position of the points involved in the formula.

5.1 Finite difference equations

As an example, we illustrate the one-dimensional wave equation following Delsanto [49]

\[ Eu'' = \rho \ddot{u} . \]  

(28)

Here, \( E \) denotes the Young’s modulus, \( \rho \) is the mass density and \( u = u(x,t) \) denotes the function of displacements. In principle, all variables can depend on the spatial variable \( x \) and even on the time variable \( t \). The propagation path is divided into \( N \) cells. The cell length is given as \( l_i = 1/N \) where \( l \) is the length of the structure. A finite difference approximation is used, for both the time and space derivatives, to solve Eq. (28). For small values of \( l_i \), the derivative in \( x = 0 \) of an arbitrary function \( f(x) \) takes the following form

\[ f'(0) = \frac{f(l_i) - f(-l_i)}{2l_i} - \frac{1}{3!} f'''(0) + ... \]  

(29)

A similar expansion is also employed for the second derivatives. Neglecting the powers of \( l_i \) higher than the first, Eq. (28) becomes

\[ u_{i+1} = c(u_{i+1,j} + u_{i-1,j}) + 2(1-c)u_i - u_{i-1,j} , \]  

(30)

where \( u_{i,j} = u(x = il/N, t = j\tau) \) is the discrete displacement at point \( i \) and at the time step \( j \). That is to say, the continuous time variable denoted by \( t \) is replaced by a discrete set of points in time \( j\tau \), with \( j = 0, 1, 2, \ldots \) where \( \tau \) is a defined time step. The parameter \( c \) is

\[ c = c_p/c_c \quad \text{with} \quad c_p = \sqrt{E/\rho} \quad \text{and} \quad c_c = l_c/\delta . \]

Delsanto noted that from the theory of FD equations it is known that the best choice for \( c \) is \( c = 1 \) since, for this value of \( c \), the region of determination for the FD equation coincides with the region of determination of the differential equation [49]. Utilizing \( c = 1 \), Eq. (30) is

\[ u_{i+1} = u_{i+1,j} + u_{i-1,j} - u_{i-1,j} \]  

(31)

For an extension of the finite difference schemes to the two- or three-dimensional case and application examples, we refer the interested reader to [47, 48, 168] and the references cited therein.

5.2 Extension of the finite difference method to the local interaction simulation approach

The main disadvantage of the FD schemes is that stiffness jumps due to changes in material properties cause stability problems [48]. With this in mind, Delsanto et al. developed the local interaction simulation approach (LISA) in combination with the sharp interface model (SIM) [47–49].
This approach is highly parallelizable and has been used for calculations on connected machines. Lee et al. state [141]: "The LISA is formally similar to the FD approach. However, the associated iteration equations are obtained directly from heuristic considerations. Since local interactions between elements are transferred directly for numerical calculations, the wave partial differential equations are bypassed and the algorithm is extremely efficient from a numerical point of view. In addition the LISA already inherits the features of the SIM, which lowers the number of interface grid points in heterogeneous media. The basic idea of the SIM is to average the interface grid point with the surrounding grid points. The main assumption here is that perfect/smooth contact is maintained between different material layers. The LISA/SIM allows for a more physical and unambiguous treatment of interface discontinuities for different physical layers." The SIM takes an interface of two layers with different impedance $Z_1$ and $Z_2$ into consideration. It is possible to find an iteration equation for the interface node point

\begin{equation}
\begin{aligned}
u_{j+1} = t_2^I u_{j+1} + t_1^I u_j - u_{j-1}
\end{aligned}
\end{equation}

with the transmission coefficients $t_1^I$ and $t_2^I$

\begin{align}
t_1^I &= \frac{2}{1+\zeta} = 1 + r_l, \quad \zeta = \frac{Z_1}{Z_2}, \\
t_2^I &= \frac{1}{1+\zeta} = 1 - r_l
\end{align}

and the reflection coefficient $r_l$

\begin{equation}
r_l = Z_1 - Z_2
\end{equation}

Delsanto states [49]: "It is possible, though not as immediately transparent, to give a FD justification of Eq. (32). The boundary conditions at the interface require the continuity of both the displacements and the stresses. To impose the boundary conditions, we look at two points P and Q on the two sides of the interface and infinitely close to it. As a part of the initial conditions, a unique value $u_i$ is prescribed to the displacement at the interface at the initial times $t = 0$ and $t = 1$." In other words, the continuity of the displacement is ensured by imposing equal accelerations $\ddot{u}_P = \ddot{u}_Q$ and interface stresses $\tau_P = \tau_Q$.

6 Finite element method

The FEM is a well-established numerical approach for the solution of partial differential equations in various engineering branches. In general terms, it is not possible to find the solution to a system of partial differential equations describing the problem in the whole domain, so a distinctive feature of the FEM is the division of a given domain into a set of small sub-domains, called finite elements (FEs). When higher order FE approaches are considered, the spectral element method (SEM) has been used almost exclusively for high-frequency wave propagation problems. SEM approaches are generally based on Lagrange polynomials through Gauss-Lobatto-Legendre (GLL) or Chebyshev-Gauss-Lobatto (CGL) points. In 2005 the concept of the so-called "isogeometric analysis" (IGA) was defined. The work was motivated by closing the existing gap between the finite element analysis (FEA) and computer aided-design (CAD). Using a NURBS-based description of the geometrical model in CAD reduces the effort of the discretization process.

This paper puts forward the notion that, independent of the proposed shape functions, all methods adhering to the aforementioned features should be referred to as FEM. SEM, IGA and p-FEM are merely specialized versions of the conven-
tional FEM. The following section briefly recalls the basic equations for developing the FE formulations. This is also the point of departure to derive the different versions of the FEM.

6.1 Finite element equations

The basis of the FEM is the variational formulation corresponding to Navier’s equations, namely Hamilton’s principle. It states that the motion of the system within the time interval \([t_1, t_2]\) is such that the Hamiltonian action \(S\) reaches a minimum, i.e. the motion of the system takes the path of the stationary action [92]

\[
\delta S = \delta \int_{t_1}^{t_2} (L + W) \, dt = 0 .
\]  

(36)

Here \(L\) represents the Lagrangian function, and \(W\) the work done by the external forces. The Lagrangian is the sum of the kinetic energy and the potential strain energy. The derivation of the FE equations for an elastic material is briefly explained in the next paragraphs. Following some calculations and the substitution of Hooke’s law \(\sigma = C \varepsilon\) into Eq. (36), we obtain

\[
0 = - \int \left[ \rho \delta u^T \ddot{u} + \delta \varepsilon^T C \varepsilon \right] \, dV + \int \rho \delta u^T F_v \, dV + \int \delta \varepsilon^T F_v \, dV + \sum_{i=1}^{n} \delta \varepsilon_i^T F_i \delta \varepsilon \]

(37)

where \(\rho\) is the mass density, \(\varepsilon\) and \(\sigma\) are the vectors of the mechanical strains and stresses in Voigt-notation, respectively [280]. \(C\) denotes the elasticity matrix and \(\varepsilon\) stands for the acceleration vector and \(F_v\), \(F_S\) and \(F_i\) are the vectors of volume, surface and point force, respectively. The displacement field \(u(x,t)\) is approximated by the product of the space-dependent shape function matrix \(N(x)\) and a time-dependent vector of unknowns \(U(t)\).

\[
u(x,t) \approx N(x)U(t).
\]  

(38)

The mechanical strain is defined as

\[
\varepsilon = \varepsilon N U = B U,
\]  

(39)

where \(B = \varepsilon N\) and \(\varepsilon\) is the strain-displacement-matrix. With the aid of Eqs. (37), (39) and (38) and by imposing that Hamilton’s principle has to be satisfied for all variations \(\delta \ddot{u} = \delta N \ddot{U}\), the FEM equations of motion are obtained

\[
\rho \begin{bmatrix} \int N^T \ddot{u} \, dV + \int B^T C \ddot{u} \, dV \end{bmatrix} \begin{bmatrix} v \\ M \end{bmatrix} + \int S^T dS + \int F_p \, dS = \int F_v \, dV + \int N^T F_v \, dS + \int N^T F_p \, dS
\]  

(41)

Introducing the mass matrix \(M\), the stiffness matrix \(K\) and the load vector \(F\), Eq. (41) reads

\[
M \ddot{U} + KU = F.
\]  

(42)

For further explanations, such as how to include the influence of damping in the FEM, we refer the reader to standard text books on this subject by Bathe [20], Hughes [105] and Zienkiewicz [280], for instance.

6.2 Element types

Two element types are usually considered to model thin-walled structures within the FE framework: solid and shell elements. Every degree of freedom in a solid element (cf. Fig. 5a) corresponds to the displacements in \(x_1\)-, \(x_2\)- or \(x_3\)-direction at a specific node. A finite element model (cf. Fig. 5b), on the other hand, requires a lower number of nodes per element. Here the displacement at the center plane of the structure and two rotations are used to describe the deformation state. To this end, five degrees of freedom have to be taken into account for each node.

When calculating Lamb wave propagation shell elements are limited, because they only provide accurate results when wave propagation problems with low \(f \cdot h\)-values are dealt with. Higher order modes are not fully resolved because of the assumptions made regarding the displacement field in thickness direction. Furthermore, the symmetric mode can only be shown in an indirect way, because it is not possible to describe any tension in the thickness direction. To this end, five degrees of freedom have to be taken into account for each node.

A model using shell elements requires fewer degrees of freedom than a model based on solid elements. Since the thickness direction is not explicitly discretized, shell elements are computationally less time-consuming. All physical behavior which is thickness independent can be described correctly. On the other hand, it is not possible to provide accurate numerical simulations by modeling complex structural features such as particle reinforcements and delaminations by deploying shell elements. The use of solid elements is recommended if higher order Lamb wave modes should be simulated accurately or if internal defects are of interest. Both types of elements, however, can suffer from locking phenomena related to an insufficient discretization. The following
6.3 Low order finite elements

In commercial FEM software packages low order FEs (lo-FEs), also known as \( h \)-version FEs, are almost exclusively implemented. Low order means that the polynomial degree \( p \) of the shape functions is 1 or 2. The \( h \)-version FEs are usually based on nodal shape functions. So it is easy to achieve a diagonalization of the mass matrix by means of simple algorithms, such as the row-sum technique \([94]\). The error introduced by mass-lumping is in most cases negligible \([43–45]\) and this procedure facilitates the use of explicit time-stepping methods. Because of the diagonal mass matrix, the solution to the system of equations is trivial and consists only of matrix and vector operations at the element level.

Lo-FEs are known to be very sensitive to element distortion. High values of the ratio of the longest to the shortest element edge (aspect ratio) cause the accuracy to deteriorate. The optimum degree of accuracy is reached when this ratio is equal to one. If the discretization in the thickness direction of thin-walled structures is too coarse, so that the displacement field cannot be modeled accurately, stiffening effects may be encountered. These phenomena are typically referred to as locking. When dealing with wave propagation problems, this effect results in an overestimation of the propagation velocity, which is an important characteristic. To reduce the effect of locking a reduced or selectively reduced element integration rule is often proposed in the literature. By introducing this under-integration, the stiffness matrix can become rank deficient which can cause problems like hourglassing also known as zero energy modes. To mitigate the effects of hourglassing several algorithms have been proposed \([105]\). The influence of hourglass control schemes on the accuracy of wave propagation analysis has been illustrated in Willberg et al. \([271]\). The results, depicted in Fig. 6, were obtained using the default control mechanism employed by the commercial software package ABAQUUS (originally proposed by Flanagan and Belytschko \([71]\)). Despite their shortcomings lo-FEs are often used for wave propagation analysis. Explicit time-integration is used by Bartoli et al. to compute the propagation of guided waves in railroad tracks \([18]\). Gresil et al. \([85]\) investigated the influence of corrosion on Lamb waves. Greve et al. \([86]\) studied the transition from Lamb waves to longitudinal waves in thick plates numerically and experimentally. Bijudas et al. \([24]\) conducted experimental and numerical studies of “baseline-free” damage detection in a stiffened plate by time-reversed Lamb waves. Vanli and Jung \([254]\) employed \( h \)-version FEs in conjunction with statistical updating methods to improve the damage prediction capability. In \([161]\) wave scattering at impact damages was investigated in sandwich panels. Sause et al. \([217]\) used the FEM to model Lamb wave propagation in anisotropic composite materials, the special focus being put on pressure vessels. A different application is presented in \([114]\). Here, both mode selection and steering of guided waves are accomplished and controlled electronically by means of a phased array approach. Rogge and Leckey \([214]\) presented a local Fourier domain method for processing guided wavefield data. This approach is used to estimate spatially dependent wavenumber values, which can be employed to determine the delamination depth. Regarding the quality of the results 10 - 20 nodes per wave-
length are generally recommended for an appropriate solution accuracy [85]. Depending on the accuracy requirements, however, this is not always true. To illustrate the effect of the discretization on the group velocity of the wave packet, Figs. 6 and 7 show the relation between the relative error in the calculated time-of-flight and number of nodes per symmetric and anti-symmetric wavelength, respectively. It is evident that a very fine spatial discretization is needed if very accurate results are required. From the general trend of the two dashed curves, we infer that the fundamental symmetric mode is relatively insensitive towards element distortion, whereas the results for the fundamental anti-symmetric mode are notably influenced by a low aspect ratio (≈1). Along the same lines, we observe that a selectively reduced integration only improves the accuracy of the anti-symmetric Lamb wave mode. This can be explained by the fact that the $S_0$-mode essentially resembles an extensional-compressional wave, while the $A_0$-mode resembles a flexural wave. An increased mesh density is therefore more important for resolving the bending behavior. The solid lines in Fig. 6 illustrates a potentially deteriorating effect on the accuracy of the results caused by wrong hourglass control algorithms. Fig. 7, on the other hand, clearly illustrates the influence of the locking effect. This phenomenon causes the reduced accuracy visible between the two dashed curves. Not only the spatial discretization but also the temporal one needs to be quite fine to accurately solve the wave equation for ultrasonic guided waves. There are several ways of reducing the computational effort, the first one being to reduce the size of the model, thus decreasing the number of degrees of freedom. In this regard, Oh et al. [185] developed non-reflecting boundary conditions and applied them to a concrete structure. Hosseini et al. [99] adopted a novel approach based on dashpot element and used carbon fiber-reinforced plastic (CFRP) and honeycomb sandwich plates to minimize the effect of wave reflections at the boundaries. The advantage of the method proposed by Hosseini is that it can be readily implemented in commercial software packages. A non-reflecting boundary is introduced by Liu and Jerry [151], thus simulating an unbounded domain while maintaining a finite computational domain. Liu and Jerry states [151] that “no spurious reflections are caused at the computational boundary. Progressive damping is introduced to a section of elements before the finite boundary of the finite element model, thus minimizing reflections of the wave, if not damping them down completely. The damping near the boundary is applied by following an exponentially increasing function to achieve just the required amount.” A different but also commonly used approach is the so-called perfectly matched layer (PML) method. Duru and Kreiss [65] extended the PML formulation to damp back-propagating waves that can lead to temporally growing solutions in the damping layer if the standard version of PML is adopted. For further information on non-reflecting boundary condition, we refer the reader to the listed references and the works cited therein. The methods described above, however, reduce the numerical effort of FE-simulations considerably and can be of great value to the analyst.

### 6.4 High order finite elements

Section 6.3 discussed the advantages and disadvantages of low-order FEs. A way of circumventing the said shortcomings is to employ high order FEMs (ho-FEMs). Accordingly, the focus of current research activities related to numerical wave propagation analysis based on the FEM has shifted towards ho-FEMs. The individual methods only differ in their choice of shape functions. A wide variety of shape functions has been proposed in the current literature. One possibility is to adopt Lagrange interpolation polynomials as Ansatz functions resulting in the SEM. The majority of applications are based on a non-equidistant nodal distribution. The Lagrange polynomials are often defined on a Gauss-Lobatto-Legendre (GLL) or Chebyshev-Gauss-Lobatto (CGL) grid [122,123,126,127,153,177,188,189,193,194,215,229,238]. Another suitable choice consists of hierarchical Ansatz functions based on the normalized integrals of the Legendre polynomials [50,51,66,237,243,244]. This method is commonly referred to as the $p$-version of the FEM. A recently published approach employs non-uniform rational B-splines (NURBS) as shape functions [37,54,106,107,270,270]. The term IGA was coined to refer to this method, as the element Ansatz functions are also used to describe the geometry of the structure.

In the context of ho-FEMs and the simulation of thin-walled structures, the analyst is faced with the decision whether to use shell or solid elements based on Ansatz functions where the polynomial degree is different in each coordinate direction. Depending on the intended application (excitation frequency, geometric dimensions, boundary conditions etc.) each approach has its advantages and disadvantages. Both options are discussed in the following sections.

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![Fig. 7. Convergence curve for the $A_0$-mode taken from Willberg et al. [271].](image-url)
6.4.1 High order shell elements

When dealing with thin-walled structures, one choice is to employ finite shell elements to discretize the structure. As has been pointed out in Section 6.2, this element type offers certain computational advantages over solid FEs. On the other hand, they are only able to describe the wave propagation in a limited frequency range. Detailed information about the range of applicability of spectral shell elements (SSEs) are found in the monograph by Ostachowicz et al. [164]. Various theories such as the Reissner-Mindlin plate theory, also known as the first order shear deformation theory (FSDT) and high order theories are discussed in detail. There are several research groups that have developed shell elements despite of the mentioned shortcomings. They are commonly based on the FSDT and deploy Lagrange polynomials as shape functions. Notable contributions in the context of ultrasonic wave propagation analysis have been made by Ostachowicz et al. [55, 56, 131, 132, 134, 189] and Fritzen et al. [72, 223, 224]. Ostachowicz et al. [189] deployed the SEM to simulate the Lamb wave propagation in composite plates for aeronautical applications. Its applicability and efficiency for SHM applications have been demonstrated in various publications [57, 58, 131–134]. Kudela et al. [131, 132] employed a 36-node SSE to simulate the A0-mode propagation for various orientations and relative volume fractions of a fiber-reinforced composite plate. The angle-dependent properties of the wavefield have also been highlighted. In [134] the feasibility of SSEs for damage detection applications, using an algorithm based on piezoelectric transducers arranged in a clock-like configuration, is investigated. Employing different damage maps, the authors are able to locate cracks in composite laminates. Zak et al. proposed different damage indicators (integral mean value - IMV; root mean square - RMS) and tested them in applications on aircraft structures. Numerical investigations were performed on an aluminum plate and wing showing successful damage localization. Experimental studies on a composite stabilizer of a PZL W-3A helicopter highlighted the limitations of this approach. Applying SSEs to wind turbine plates is demonstrated in [56]. Fritzen et al. [72] introduced delamination-type damages in flat CFRP shell structures, where the delamination is simply modeled by node separation in conjunction with a simple contact algorithm. The numerical results were subsequently validated using experimental data. Simulation results are also obtained for composite plates stiffened by T-strings. Schulte and Fritzen extended the formulations proposed in their article by adding damping and piezoelectric coupling [223, 224]. They accordingly included the effect of direction-dependent attenuation and were able to model the excitation of waves using piezoelectric transducers, making it possible to carry out more realistic simulations. To the best of the authors’ knowledge, shell elements based on the p-version of the FEM or the IGA have not been used for ultrasonic guided wave propagation analysis in the context of SHM applications.

6.4.2 High order solid/continuum elements

In recent years, carbon fiber reinforced plastics (CFRP), glass fiber reinforced plastics (GFRP) and sandwich composites are materials commonly used in the aircraft industry. To model the interaction of the wave field with the microstructure of the medium solid elements are needed. It is only then that, effects like material-induced mode conversion [273], or wave scattering at the core layer of sandwich panels [101], and many others can be properly investigated. Continuum FEs consequently offer a broader variety of applications [59]. Two examples where the discretization over the thickness of the plate is essential for accurately modeling Lamb wave propagation problems are illustrated in in Fig. 8. The first example - cf. Fig. 8a - illustrates the mode conversion at a geometrical perturbation of the plate, whereas the second example - cf. Fig. 8b - depicts the propagation of the A1-mode.

The following paragraphs discuss important contributions dealing with wave propagation analysis using different high-order FEs. The SEM is to date the most used method to deal with high frequency wave propagation problems. The p-version of the FEM and the IGA have been used mainly for static or modal analyses [68, 102, 219, 221]. Willberg et al. [272] used the SEM, the p-FEM and the IGA to solve ultrasonic guide wave propagation problems. The published results show that the convergence characteristics of all high order C0-continuous FE schemes are very similar. A guideline on how to choose the spatial discretization according to the polynomial degree of the shape functions is derived from the convergence curves and shows that it is possible to reduce the numerical costs compared to lo-FE approaches. Willberg [272] and Duczek [59, 60] also introduced coupled-field elements for the IGA and the p-FEM in order to model wave excitation via piezoelectric transducers. De Basabe and Sen made an important contribution to the field of wave propagation analysis concluding that the dispersion results gathered for the acoustic wave propagation can be easily transferred to the elastic case [46]. Dauksher and Emery arrive at the conclusion that the behavior of high order methods is very similar for both wave equations [45]. These results were obtained by studying wave propagation problems with spectral elements (SEs). For this reason, the following paragraphs also contain contributions from the field of seismic wave propagation, since these results are also applicable to guided wave propagation problems. Seriani and Su [227, 231] demonstrated the versatility of the SEM for wave propagation purposes. They solved the acoustic wave equation for complex media (seismic wave propagation in the earth’s mantle) using a SEM poly-grid approach. By introducing temporary auxiliary grids to deal with a continuous variation in physical properties, they avoid the need to employ a fine global grid. This approach bears similarities to the methodology adopted by fictitious domain methods [61]. Seriani and Priolo found that the SEM shows more accurate results than h-version FEs [230]. From their research they concluded that there is a theoretical limit to the number of nodes per minimum wavelength of at least \( \pi \) for an increasing polynomial degree. This result is confirmed...
by Ainsworth [10,11]. Using this kind of discretization, the accuracy remains almost unchanged for long propagation times. For practical applications, however, they recommend 4.5 nodes per wavelength using a polynomial degree of $p = 8$. Seriani and Oliveira [229] studied SEs based on a GLL and CGL grid for wave propagation problems. They observed the following properties; SEs using a CGL grid exhibit a phase lead, while GLL based SEs produce lagging errors. This was also noted in references [12, 247] for lumped and consistent mass matrix formulations. Peng et al. utilized three-dimensional continuum SEs to study the effect of the adhesive layer on the wave propagation and to show the excellent properties of SEM for damage detection in laminates and beams [148,194–196]. Komatitsch and co-workers first applied the SEM to problems concerning seismic wave propagation [124,125,128]. A fully three-dimensional earth model is used to simulate global seismic wave propagation problems. The influence of the rotation, self-gravitation and the oceans is taken into account, whereas the oceans are modeled as an equivalent load. Ha and Chang developed a so-called hybrid SE [88–90]. For this type, the polynomial degree in out-of-plane direction of the plate is set to $p = 1$, whereas the in-plane polynomial degree is arbitrary. Using this approach, they studied the effect of the adhesive layer between piezoelectric transducer and host structure. They found that the resonance effect affects signal amplitudes in a manner opposite to shear lag, which is a dominant effect in frequencies far removed from the resonance.

The widely accepted use of the SEM in dynamic problems is due to the fact that mass lumping is easy to achieve. Dauksher and Emery [43,44] demonstrated that a row-summing procedure can result in accurate solutions with SEs. This row-summing procedure is only performed on the global mass matrix. Nonetheless, the solution characteristics are relatively unaffected by the diagonalized mass matrix, but the numerical costs for the solution of the equation system are drastically reduced when applying an explicit time-integration scheme. The authors opt for deploying SEs with a row-summed mass matrix in conjunction with the central difference method (CDM) to solve the wave equation. Another possibility is to use the same nodes, on which the shape functions for SEM have been defined, as points for a Gaussian quadrature. SEs based on a Gauss-Lobatto-Legendre (GLL) grid coupled with the well-known GLL-quadrature rule provide a diagonal the mass-matrix [272]. Jensen [111], Christon [35] and Ainsworth and Wajid [11,12] comment on the influence of the mass matrix on the numerical results. Jensen [111] employs the SEM and concludes that the diagonal mass matrix obtained with Gauss-Lobatto-Legendre quadrature does not introduce significant errors.

There is a wealth of literature on error estimates, error indicators and the reduction of numerical dispersion, particularly in the context of the $p$-version of the FEM. For detailed information, we refer the reader to [109,110,226,228,229] and the references cited therein.

A different approach aimed at increasing the accuracy of the simulation is to manipulate the consistent mass matrix. Christon [35] studies the dependence of the dispersive error on the wave propagation direction, mesh aspect ratio and wavenumber. This article shows that a high order mass matrix - a linear combination of lumped and consistent mass matrices - may improve the dispersion characteristics of both reduced and fully integrated FE. Ainsworth and Wajid [11,12] also utilized a weighted average of the consistent and diagonalized mass matrices. They demonstrate that the optimal blending parameter for FE of order $p$ is given by the ratio $1 : p$. Compared to the pure schemes, it is possible to gain an additional two orders of accuracy, regardless of the spatial dimensions.

6.5 The spectral element method in the frequency domain

The flexibility of FEMs to model a variety of geometries makes them an invaluable technique for modeling wave propagation in complex structures. The process of wave propagation involves the study of the evolution of displacements through time. In the classical FEM, this is
done with the help of a time integration scheme and it is said that the calculation takes place in the time domain. An alternative method is to transform the FE equations to the frequency domain. The aforementioned method has already been in use for several years for the purpose of modal vibrational analysis and is known as the dynamic stiffness method [15, 145–147, 213].

Modal vibrational analysis is a conventional method for computing the dynamic response of a structure when the frequency content of the input excitation is not very high. For wave propagation problems where the excitation signals possess a high frequency content, the wavelengths are accordingly very small. In these cases, the FE meshes should be very fine and many eigenmodes need to be calculated in order to accurately model wave propagation, which involves a great deal of computational effort. On the other hand, using a numerical time marching scheme is not advantageous for these very fine meshes, because the restrictions on the time-step render the wave propagation simulation unfeasible.

Several approaches have been developed to overcome this problematic issue. One tendency is to use shape functions that model the geometry more appropriately. This opens up the possibility of taking bigger FE to reach a certain degree of accuracy while simultaneously making the time integration process more feasible. Elements developed according to this principle are known as super convergent elements [28, 29, 80, 174]. The difficulty with these methods is that they were developed for specific problems and specific structures.

An alternative proposal is to solve the problem analytically in the frequency domain. This can be regraded as a completely different approach, since it involves removing the time variable from the equation and introducing the frequency as a parameter. For this reason, it is necessary to solve a time-independent problem for each frequency. These elements are also called SEs when the Fourier transform is used for this purpose [81]. In addition, wavelet and Laplace transforms have been used to explore the advantages of other domains besides the frequency domain [82] in order to avoid the "wrap around" problem that arises when modeling bounded spatial domains with the Fourier transform.

The same principle can be applied to bounded domains when material dissipation is included in the model or absorbing boundary conditions are present in the problem. In this particular case, it is possible to combine the discretization of SEs in time domain and the SEM in the frequency domain, in order to increase the accuracy [155, 257].

### 6.6 Assessment of finite element methods

ho-FEMs are generally more accurate than lo-FEMs [271]. These methods do not only lead to a significant reduction in memory requirements but also minimize the numerical dispersion error [43–45]. Besides, the computational costs are also considerably lower. A second advantage is that ho-FEMs may display an exponential decay of the error in the energy norm [66], thus overcoming the poor convergence rates that occur with lower order FEM [243, 244]. There are basically three ways of controlling the error:

1. decreasing the element size (h-refinement),
2. increasing the polynomial degree of the Ansatz functions (p-refinement),
3. increasing the degree of the inter-element continuity (k-refinement).

and combinations of those. A k-refinement has proved to be very efficient for ultrasonic guided wave propagation [271]. The authors, however, favor the solid element formulation over the shell one. Due to the assumptions used to derive a shell theory the propagation of symmetric and anti-symmetric Lamb wave modes can only be described within a certain frequency range [189, 224]. Another drawback is that multi-layered materials and complex three-dimensional stress states arising at welded joints or rivets, for example, cannot be resolved. Consequently, approaches based on utilizing high order shell elements are not appropriate if all phenomena that occur when dealing with wave propagation problems need to be taken into account. Solid volume elements in the other hand, although being computationally more expensive, provide the opportunity to model all different kinds of real-life structures and is therefore not restricted to certain modeling assumptions. It is consequently well suited for computing the full wave field of complex structures. When dealing with physical dispersion, however, it is advisable to employ computational schemes such as the SAFE method or purely analytical approaches.

The IGA bridges the gap between CAD and FEA [21, 22, 37, 38, 106] as it deploys the functional description of the geometry from the CAD software (B-splines, NURBS, T-splines, etc.) and reuses them as shape functions for the FEA [37]. So no discretization process is needed to approximates the CAD geometry and the exact description of the geometry is recovered. This is an important consideration for shape optimization schemes. Not only is the exchange between the FE software and the CAD software improved, but also the exact CAD geometry is used for the optimization [202].

By way of using the SEM the degrees of freedom of nodal-based FE schemes retain their physical meaning so it is possible to employ standard mass-lumping techniques. It is consequently possible to exploit the advantages offered by explicit time-stepping schemes. This results in significant savings in terms of computational time and memory requirements.

The main advantage when using the p-FEM is the hierarchic structure of its system matrices. This property facilitates the implementation of adaptive refinement techniques [23, 66]. Since adaptive refinement is beyond the scope of this treatise the reader is referred to [50, 51, 237].

### 7 Other methods

Besides the methods that have been presented, several other numerical approaches have also been used in the context of wave propagation analysis. In the following para-
graphs a brief description of these methods is presented. Yim et al. [277] used a mass-spring lattice model (MSLM). The advantage of this model is the low computational effort compared to FEM or other non-analytical methods. However, the MSLM material parameters have no physical meaning, because in complex structures a local spring stiffness is not measurable.

Su and Wang coupled FEs with infinite elements to model plates of virtually infinite lengths and avoid reflection from boundaries [240]. Düster and Parvian [69, 192] introduced the finite cell method (FCM) and several extensions were presented by Schillinger et al. [220]. This approach is a combination of \textit{ho}-FEMs and the fictitious domain concept. High rates of convergence, possibly even exponential, can be achieved using this approach because of the polynomial Ansatz functions of arbitrary degree. Employing the fictitious domain idea an automated discretization is provided. Initial studies of the FCM in the field of wave propagation analysis were conducted by Duczek et al. [59, 61, 62, 113]. Duczek introduced spectral shape functions in the framework of the FCM developing the spectral cell method (SCM) [61]. Using the SCM explicit time-integration methods such as the CDM can be exploited to speed up the simulation. Multi-physics applications are also possible. Both electro-mechanic and thermo-mechanic versions of this scheme were already developed [62, 278]. The published results highlight that this approach could be especially interesting in the context of wave propagation analysis in thin-walled structures. Leckey et al. [139] used an elastodynamic finite integration technique (EFIT) approach for wave propagation analysis in anisotropic and quasi-isotropic composite structures from the aerospace industry.

Ham and Bathe developed an enriched FEM [91]: This method enriches the elements with extra trigonometric functions to reach a higher performance in the simulation process and avoid numerical dispersion effects. It is shown that the error between a 4-node traditional linear element and an enriched element is drastically reduced. To avoid the limitations of normal finite plate elements based on the Kirchhoff assumptions Yang et al. [276] studied plate elements deploying a higher order theory. Although the mentioned methods are also interesting they are currently used by only a few researchers and are therefore not considered as established methods.

8 Conclusion

Depending on the intended application, each particular method has its advantages and disadvantages. It is therefore impossible to state which of those schemes is the most efficient without taking the area of application into account. The following paragraphs are intended as a summary and to give a brief overview of the most characteristic features of the reviewed methods. Analytical and semi-analytical methods are particularly suitable for the computation of dispersion curves and the wave propagation behavior in the frequency domain. The first approach is commonly used for homogeneous materials, while semi-analytical methods are widely employed for laminates. In this context, both methods are much faster at obtaining the dispersion characteristics than all other methods described in this paper. They are also faster and more efficient in terms of accuracy when modeling wave propagation at a few points in comparison to all methods analyzed in this review. Due to their inability to simulate complex structures the use of these methods is limited to structures with relatively simple geometries. Although some effort has been directed towards the application of these methods in the modeling wave propagation in the presence of complex damage scenarios or complex geometries of the waveguide, all attempts converge towards the combination of analytical or semi-analytical methods with other numerical methods in the frequency domain. Finite difference methods and their variants such as LISA are easily parallelizable. Since only vector-matrix operations are needed to integrate the equations of motion, the time-stepping scheme can be implemented straightforwardly on graphic cards (CUDA). As long as the system matrices are constant and no communication between the CPU and GPU is required, there is a significant speed-up in computation time. This is essentially true for linear problems. Accurate solutions call for a fine discretization, especially if complex geometries are taken into account. Geometries with surfaces and lines that are curved or skew require a very fine point grid if pure finite difference methods are used. In this cases, special methods such as cellular automata, or coordinate transformations, for instance, should be taken into account in order to deal with this type of geometries. The availability of commercial tools and the flexibility to deal with arbitrary geometries is the main advantage of the FEA. Advanced pre- and post-processing tools are readily available. Linear FEs are an important part of every commercial FE software and therefore validated and robust algorithms exist. Unfortunately, \textit{h}-version FE codes are not optimized to handle ultrasonic guided wave propagation and a very fine spatial resolution is needed to approximate the geometry appropriately. To reach a certain level of accuracy the number of degrees of freedom is significantly increased in comparison to \textit{ho}-FEMs. Both shell and continuum elements are possible options when guided wave propagation simulations must be carried out. Shell elements are a good choice if simulations are performed in the lower frequency range and it is not required to model the wave propagation behavior in the thickness direction. High order solid elements do not have that limitation. Their efficiency can be improved if the polynomial degree in the thickness direction is lower than that in the propagation direction. Consequently, we recommend to use an anisotropic Ansatz space. Regarding the SEM the computational time is drastically decreased, compared to other \textit{ho}-FE approaches, if the mass matrix is diagonalized. This is also advantageous for the implementation of highly parallel algorithms. The flexibility of \textit{ho}-FEMs comes at the cost of an increased computational time compared to shell elements or analytical methods. Spectral methods in the frequency domain have the advan-
tage of being computationally faster than other methods in time domain, if wave propagation has to be modeled only in certain parts of the structure. They are accordingly very efficient for simulating the wave propagation in actuator sensor networks. Nonlinear effects or the calculation of amplitudes at a number of different points are more expensive from the numerical point of view.

Besides analytical, semi-analytical, FDMS and FEMs other methods are also used in wave propagation analysis. A short overview of these methods was provided in the previous section. These are only specialized methods that are yet fully developed. Research regarding their convergence properties and efficiency is still pending.

The authors hope that the recent literature review will help to facilitate the choice of suitable numerical methods for various problems regarding wave propagation analysis.

Acknowledgements

The second author would like to thank the German Research Foundation (DFG) and all project partners (PAK 357) for their support (GA 480/13-3). The third author acknowledges the support received from the Transfer Center MRO. We also express our sincere gratitude to Prof. Chin An Tan and the anonymous referees for their valuable tips and suggestions, which considerably improved the first version of the present paper.

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