Anisotropic hierarchic finite elements for the simulation of piezoelectric smart structures

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Abstract
Purpose – Piezoelectric actuators and sensors are an invaluable part of lightweight designs for several reasons. They can either be used in noise cancellation devices as thin-walled structures are prone to acoustic emissions, or in shape control approaches to suppress unwanted vibrations. Also in Lamb wave based health monitoring systems piezoelectric patches are applied to excite and to receive ultrasonic waves. The purpose of this paper is to develop a higher order finite element with piezoelectric capabilities in order to simulate smart structures efficiently.

Design/methodology/approach – In the paper the development of a new fully three-dimensional piezoelectric hexahedral finite element based on the p-version of the finite element method (FEM) is presented. Hierarchic Legendre polynomials in combination with an anisotropic ansatz space are utilized to derive an electro-mechanically coupled element. This results in a reduced numerical effort. The suitability of the proposed element is demonstrated using various static and dynamic test examples.

Findings – In the current contribution it is shown that higher order coupled-field finite elements hold several advantages for smart structure applications. All numerical examples have been found to agree well with previously published results. Furthermore, it is demonstrated that accurate results can be obtained with far fewer degrees of freedom compared to conventional low order finite element approaches. Thus, the proposed finite element can lead to a significant reduction in the overall numerical costs.

Originality/value – To the best of the author’s knowledge, no piezoelectric finite element based on the hierarchical-finite-element-method has yet been published in the literature. Thus, the proposed finite element is a step towards a holistic numerical treatment of structural health monitoring (SHM) related problems using p-version finite elements.

Keywords Piezoelectricity, Vibration, Noise control, Higher order finite elements, p-version FEM, Smart structures, Structural health monitoring

Paper type Research paper

1. Introduction
The concept of smart (adaptive, intelligent) structures receives an increasing attention in many branches of engineering. By deploying smart materials one has the chance to actively alter the structural response of a system and is able to adjust the design to different requirements (Büter and Hanselka, 2004). Important areas of application are shape control (Marinković, 2007) and active vibration damping (Ringwelski, 2011).

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In several industries, especially in aeronautics and aerospace, the interest in structural health monitoring (SHM) applications using piezoelectric materials as distributed sensors and actuators is steadily growing (Cuc, 2010; Giurgiutiu, 2008; Raghavan and Cesnik, 2007). Hence, when dealing with smart designs new challenges arise. For example, the optimal placement of transducers or energy supply issues has to be considered. These questions have not been on the agenda when designing only passive structures. In order to cease the opportunities presented by utilizing active shape control and vibration suppression, a versatile numerical tool to predict the structural behavior is needed. In practical applications the finite element method (FEM) provides a viable means for the modeling and the numerical analysis of engineering structures. Consequently, an efficient FEM-tool suited for smart materials ought to be developed to tackle problems arising in industrial applications. In the following it is shown that the p-version of the FEM allows for a holistic approach in modeling smart structures with high efficiency and accuracy. At least to the author’s knowledge this presents a novel approach, since no piezoelectric finite element based on hierarchical shape functions has been published in literature, yet. Due to the possibility to apply an anisotropic ansatz space, complex structures can be readily discretized using only one single element type. Both plate-like and beam-like structures as well as three-dimensional continua can be modeled with a hierarchical hexahedral finite element utilizing a variable polynomial degree template \( p_\xi = p_1, \ p_\eta = p_2, \ p_\zeta = p_3 \). This template can be adjusted to the intended application. Using a suitable polynomial degree for each direction thin-walled designs can be analyzed efficiently (Düster, 2002). For polynomial degrees of \( p > 4 \) errors caused by the so-called locking phenomenon do not pollute the solution noticeably. That is to say, for all engineering purposes the solution ought to be regarded as “locking-free”. Another advantage of higher order schemes is that the aspect ratio, longest element dimension divided by the shortest one, can be as high as 200 without inducing numerical difficulties. Such distorted elements can occur if adhesive layers are to be included in the simulation, or if thin-walled structures are examined. Hence, the proposed finite element concept is very flexible and can be applied to all problems occurring in industry. For a review of various piezoelectric finite element approaches, we refer to Köppe et al. (1998) and Marinković et al. (2006).

The objective of this paper is to develop a higher order finite element with piezoelectric capabilities in order to simulate smart structures efficiently. A higher order approach is being chosen to circumvent the problems arising in coupling different element types, such as shell or beam elements to continuum elements, on the one hand. Thus, even the most complex structures can be readily discretized using only one element type (three-dimensional continuum element) with different approximation orders. On the other hand, no effects are neglected since a fully three-dimensional model is proposed instead of a dimensionally reduced one. Despite the higher numerical costs, it is found that an efficient analysis is possible if an anisotropic polynomial degree template is utilized. Hence, the new approach presented in this article is very flexible and can be applied to a vast class of problems arising in engineering practice.

The paper is organized as follows: After defining the fundamental principles of the hierarchic FEM, the basic equations of the linear theory of piezoelectricity are discussed. Afterwards, a higher order finite element with mechanical (displacements) as well as electrical (electrical potential) degrees of freedom is developed. In the next section benchmark examples are solved in order to verify the proposed finite element scheme.
and to demonstrate its capabilities for both static and dynamic problems. Thereafter, the main ideas of the current paper are summarized and an outlook to ongoing research is given.

2. Finite-element-method and hierarchic shape functions

This section is dedicated to introduce the concept of hierarchic shape functions and the fundamentals of the p-version of the Finite-Element-Method (p-FEM). Following the approach first established by Szabó and Babuška (1991) and seized by many others Solin et al. (2004), Demkowicz (2006), Demkowicz et al. (2008), Weinberg (1996), Düster (2002), Nübel (2005), Heisserer (2008) and Becker (2007) it is shown how hierarchic basis functions can be generated up to any desired polynomial degree. Furthermore, the advantages of utilizing a hierarchic concept will be underlined. General problems arising when dealing with the p-version will be addressed as well.

2.1 Legendre polynomials

In the context of higher order FEMs orthogonal polynomials are a fundamental component. All shape functions, commonly used for higher order finite elements, are based on a set of orthogonal polynomials. The presentation of those shape functions follows (Szabó and Babuška, 1991; Düster, 2002) closely.

2.1.1 One-dimensional hierarchic basis. Shape functions based on Legendre polynomials \( L_n(x) \), a special case of the Jacobi polynomials (Solin et al., 2004), are proposed by Szabó and Babuška (1991). Legendre polynomials can be determined either by applying the Rodriguez formula:

\[
L_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} [(x^2 - 1)^n], \quad n = 1, 2, \ldots; \quad -1 \leq x \leq 1, \quad (1)
\]

or Bonnet’s recursion formula:

\[
L_{n+1}(x) = \frac{1}{(n+1)} [(2n+1)xL_n(x) - nL_{n-1}(x)], \quad n = 1, 2, 3, \ldots; \quad -1 \leq x \leq 1. \quad (2)
\]

Here the index \( n \) denotes the polynomial degree. If equation (2) is to be used to generate Legendre polynomials the recursion starts with:

\[
L_0(x) = 1, \quad (3)
\]

\[
L_1(x) = x. \quad (4)
\]

Note that all roots of the Legendre polynomials \( L_n(x) \) lie within the interval \(-1 \leq x \leq 1\) and are probably better known as Gauss points, utilized for the Gaussian quadrature. In order to construct shape functions for p-elements the normalized integrals of the Legendre polynomials \( \Phi_n(\xi) \), further on labeled modified Legendre polynomials, are used:

\[
\Phi_n(\xi) = \frac{\sqrt{2n-1}}{2} \int_{-1}^{1} L_{n-1}(x)dx = \frac{1}{\sqrt{4n-2}} [L_n(\xi) - L_{n-2}(\xi)], \quad n = 2, 3, \ldots \quad (5)
\]
Thus, the value of the modified Legendre polynomials is equal to zero at the boundaries of the considered interval \(-1 \leq \xi \leq 1\). Their derivatives, necessary to calculate the element stiffness matrix, are given by:

\[
\frac{\partial \Phi_n(\xi)}{\partial \xi} = \frac{1}{\sqrt{4n-2}} \left[ \frac{\partial L_n(\xi)}{\partial \xi} - \frac{\partial L_{n-2}(\xi)}{\partial \xi} \right], \quad n = 2, 3, \ldots
\]  

(6)

Within the framework of the hierarchic concept the linear Lagrange polynomials corresponding to the corner nodes (\(\xi = \pm 1\)) constitute the first two shape functions \(N_1(\xi)\) and \(N_2(\xi)\):

\[
N_1(\xi) = \frac{1 - \xi}{2}, \quad \text{(7a)}
\]

\[
N_2(\xi) = \frac{1 + \xi}{2}. \quad \text{(7b)}
\]

They are commonly called nodal modes or nodal shape functions. They can be thought of as being external, since they are responsible for ensuring \(C_0\)-continuity between adjacent elements. Higher order ansatz functions \(N_n(\xi)\) can be generated using equation (5):

\[
N_n(\xi) = \Phi_{n-1}(\xi), \quad n = 3, 4, \ldots, p + 1 \quad (p\text{-polynomial degree}). \quad \text{(8)}
\]

They are also referred to as internal modes or bubble modes, because they only affect the ansatz within the element domain and do not directly contribute to the solution of adjacent finite elements:

\[
N_n(\pm 1) = 0, \quad n = 3, 4, \ldots, p + 1. \quad \text{(9)}
\]

One of the most important advantages of using the introduced set of shape functions is that it is hierarchic. That is to say, all shape functions of lower order are included in the set of higher order shape functions (Figure 1). Meaning, if the shape functions for the polynomial degree \(p\) are known the set for \(p + 1\) can be created by adding just one new function to the set. This holds advantages for the system matrices and their assembling process. In contrast to the described behavior conventional finite elements based on Lagrange polynomials are not hierarchic and thus for every polynomial degree a new set of ansatz functions has to be generated ( Düster, 2002).
2.1.2 Three-dimensional hierarchic basis. The implementation of the p-version of FEM in three dimensions is based on a hexahedral element formulation, due to the higher accuracy compared to a tetrahedral approach (Becker, 2007). Furthermore, we utilize the tensor product space instead of the also widely used trunk space (Szabó and Babuška, 1991). It has been found that this type of ansatz space holds advantages with respect to ultrasonic wave propagation problems (see Section 3.4). Figure 2 shows the standard three-dimensional hexahedral element defining the node, edge and face numbers as well as the local coordinate system ($C_0$-continuity). The three-dimensional shape functions can be derived by calculating the tensor product of the one-dimensional shape functions:

$$N_{3D}^{3D}(\xi, \eta, \zeta) = N_i(\xi) \cdot N_j(\eta) \cdot N_k(\zeta),$$

where

$$i \in \{1, 2, \ldots, (p_\xi + 1)\},$$

$$j \in \{1, 2, \ldots, (p_\eta + 1)\},$$

$$k \in \{1, 2, \ldots, (p_\zeta + 1)\},$$

$$l \in \{1, 2, \ldots, (p_\xi + 1) \cdot (p_\eta + 1) \cdot (p_\zeta + 1)\}.$$ 

(10)

Generally speaking, three-dimensional shape functions can be classified into four distinct groups:

1. nodal shape functions (nodal modes);
2. edge shape functions (edge/side modes);
3. face shape functions (face modes); and
4. internal shape functions (internal/bubble modes).

There are eight nodal modes. These are the same trilinear ansatz functions used for the well known eight-noded isoparametric hexahedral element in h-version FEM computer programs:

**Figure 2.**
Standard hexahedral element defined on the interval $(-1 \leq \xi \leq 1, -1 \leq \eta \leq 1, -1 \leq \zeta \leq 1)$

**Notes:** (a) Definition of needs, edges, faces and local coordinate system; (b) definition of local edge orientation
\[ N_{(1,1,1)}^N(\xi, \eta, \zeta) = \frac{1}{8} (1 + \xi \xi)(1 + \eta \eta)(1 + \zeta \zeta), \quad i = 1, 2, \ldots, 8. \]  

\[ N_{(1,1,1)}^{E_1}(\xi, \eta, \zeta) = \frac{1}{4} (1 - \eta)(1 - \zeta) \Phi_i(\xi), \quad i = 2, 3, \ldots, p_\xi. \]  

Face modes are associated with the faces of the finite element \((p_\xi, p_\eta, p_\zeta \geq 2, \text{tensor product space})\). Face modes are generated separately for each face and vanish at all other faces. Considering face 1 the corresponding face modes read:

\[ N_{(1,1,1)}^{F_1}(\xi, \eta, \zeta) = \frac{1}{2} (1 - \zeta) \Phi_i(\xi) \Phi_j(\eta), \quad i = 2, 3, \ldots, p_\xi, \quad j = 2, 3, \ldots, p_\eta. \]  

Internal modes \((p_\xi, p_\eta, p_\zeta \geq 2, \text{tensor product space})\) are purely local and vanish at all nodes, edges and faces of the hexahedral finite element:

\[ N_{(1,1,1)}^{int}(\xi, \eta, \zeta) = \Phi_i(\xi) \Phi_j(\eta) \Phi_k(\zeta), \quad i = 2, 3, \ldots, p_\xi, \quad j = 2, 3, \ldots, p_\eta, \quad k = 2, 3, \ldots, p_\zeta. \]  

The indices \(i, j, k\) of the shape functions denote the polynomial degrees in \(\xi, \eta\) and \(\zeta\)-direction, respectively. It can be noted that the polynomial degree can be varied separately in each local direction in order to create what is called an anisotropic ansatz spaces. This is especially relevant when dealing with thin-walled structures (Düster et al., 2001).

### 2.2 Piezoelectricity

The focus of this section is on summarizing the fundamental concepts of the linear theory of piezoelectricity and on deriving a hexahedral finite element with both mechanical (displacements) and electrical (electric potential) degrees of freedom, thus, facilitating an electro-mechanical coupling.

The piezoelectric effect was first discovered in 1880 by Jacques and Pierre Curie. When subjected to a mechanical deformation, a piezoelectric material generates an electric polarization and vice versa. The first effect is called the direct piezoelectric effect and is the foundation of all sensor applications. The second effect is the ability to generate an external force proportional to the applied charge, which is accordingly called the converse (inverse) piezoelectric effect. All piezoelectric actuators are based on that principle. Essentially, the piezoelectric effect is an energy transfer between mechanical and electrical energy (Piefort, 2001).

To describe the coupled electro-mechanical field problem the mechanical equilibrium equations, balance of linear momentum (equation (15)) and balance of moment of momentum (equation (16)), as well as the electric equilibrium equations, the fourth Maxwell equation (equation (17)), are required:

\[ \text{div}(\sigma) + \dot{\rho} - \rho \ddot{u} = 0, \quad (15) \]

\[ \alpha^T = \sigma, \quad (16) \]
\[ \sigma = \{\sigma_{xx} \ \sigma_{yy} \ \sigma_{zz} \ \tau_{xy} \ \tau_{yz} \ \tau_{zx}\}^T \]

\(\text{div} (\mathbf{D}) = 0.\) \hspace{1cm} (17)

\(\sigma = \frac{\partial H}{\partial \varepsilon} = C_\varepsilon - \mathbf{e}^T \mathbf{E},\) \hspace{1cm} (18)

\(\mathbf{D} = -\frac{\partial H}{\partial \mathbf{E}} = \mathbf{e} \varepsilon + \kappa \mathbf{E}.\) \hspace{1cm} (19)

Here the mechanical strain \(\varepsilon\) and the electric field \(\mathbf{E}\) are chosen as independent variables (Ikeda, 1990; Seeger, 2003; Lefèvre, 2007). The matrix of piezoelectric coupling constants \(\mathbf{e}\) relates the mechanical stresses \(\sigma\) to the electric field \(\mathbf{E}\) when no mechanical strains \(\varepsilon\) are present within the regarded body. \(C\) denotes the Hooke's matrix of elastic coefficients for a constant electric field. Scrutinizing equation (19) we see that \(\mathbf{e}\) also relates the electric displacements \(\mathbf{D}\) with the mechanical strains under a zero electric field (short-circuited electrodes). \(\kappa\) symbolizes the matrix of dielectric constants (permittivity) for a constant mechanical strain state. Note that the constitutive matrices are not independent. The following relation between different material matrices is valid (Marinković, 2007):

\[ \mathbf{d} = \mathbf{e} \mathbf{S}^E, \quad \mathbf{S}^E = C^{-1}.\] \hspace{1cm} (20)

2.3 Piezoelectric finite element based on the p-version

The current section deals with the finite element formulation of an electro-mechanically coupled system. The most important equations are compiled and the system matrices are derived in order to be capable of simulating active structures. Thus, the static and dynamic behavior of structures containing active elements can be evaluated. Since the derivation is based on the linear theory of piezoelectricity, only small deformations and rotations as well as a low electric field is allowed. The formulation of finite elements augmented with piezoelectric properties follows the approach taken by Allik and Hughes (1970), Seeger (2003), Lefèvre (2007), Marinković (2007) and Ringwelski (2011).

The equations (15) and (17), introduced in Section 2.2 are the starting point for the formulation of piezoelectric finite elements. The principle of virtual work will be exploited to derive the system matrices and general load vector corresponding to the FEM. Using the virtual displacements \(\delta \mathbf{u}\) and the virtual electric potential \(\delta \Phi\) as weighting functions, the governing differential equations can be expressed in a weak form. Taking also the boundary conditions into consideration one obtains the following variational system of equations:

\[ \delta \chi_u = \int_V \delta \mathbf{u}^T \left( L_u^T \sigma + \mathbf{p} - \rho \ddot{\mathbf{u}} \right) dV + \int_{O_i} \delta \mathbf{u}^T (\bar{\mathbf{t}} - \mathbf{t}) dO = 0, \] \hspace{1cm} (21)
\[
\delta \mathcal{X}_\Phi = \int_V \delta \Phi \left( L^T_\Phi D \right) dV - \int_{\partial V} \delta \Phi (\bar{Q} - Q) dO = 0.
\] (22)

Equation (21) is derived from the mechanical equilibrium equation (15) and mechanical boundary conditions multiplied by a virtual displacement. Hence, the virtual work of the mechanical part is derived. Equation (22) describes the electrical part of the virtual work corresponding to equation (17) and electrical boundary conditions.

The field variables within an element \((u, \Phi)\) can be expressed by means of ansatz functions \(N_u [3 \times n_{\text{dofe}}], N_\Phi [1 \times n_{\text{dofe}}/3]\) and the degrees of freedom connected to one element \((u_e, \Phi_e)\):

\[
u = N_u u_e, \quad \Phi = N_\Phi \Phi_e.
\] (23) (24)

Substituting the constitutive equations (18) and (19), the strain-displacement relation \(\varepsilon = L_u u\), the relation between the electric field and the electric potential \(E = -L_\Phi \Phi\), the ansatz functions equations (23) and (24) into the variational formulation equations (21) and (22) yields the well-known FEM equations of motion of a piezoelectric body:

\[
0 = K^{(e)}_{uu} u_e + K^{(e)}_{u\Phi} \Phi_e + M^{(e)}_{uu} u_e - f^{(e)}_{uu},
\] (25)

\[
0 = K^{(e)}_{u\Phi} u_e - K^{(e)}_{\Phi\Phi} \Phi_e - f^{(e)}_{\Phi\Phi}.
\] (26)

Here \(L_u\) and \(L_\Phi\) are well-established differential operators. The abbreviations introduced in equations (25) and (26) denote element matrices and vectors as follows:

Element mass matrix:

\[
M^{(e)}_{uu} = \rho \int_V N^T_u N_u dV,
\] (27)

Mechanical element stiffness matrix:

\[
K^{(e)}_{uu} = \int_V B^T_u C B_u dV,
\] (28)

Direct piezoelectric element coupling matrix:

\[
K^{(e)}_{u\Phi} = \int_V B^T_u e^T B_\Phi dV,
\] (29)

Inverse piezoelectric element coupling matrix:

\[
K^{(e)}_{\Phi u} = \int_V B^T_\Phi e B_u dV,
\] (30)

Dielectric element stiffness matrix:

\[
K^{(e)}_{\Phi\Phi} = \int_V B^T_\Phi \kappa B_\Phi dV,
\] (31)
External mechanical element force vector:

\[
f^{(e)}_{uu} = \int_{V} N_{u}^{T} \Phi dV + \int_{\Gamma} N_{u}^{T} \tilde{\Phi} dO, \tag{32}\]

Electric element charge vector:

\[
f^{(e)}_{\Phi} = - \int_{\Gamma_{o}} N_{\Phi}^{T} \tilde{Q} d\Omega. \tag{33}\]

\(B_{u}\) and \(B_{\Phi}\) are the differential operator matrices applied to the ansatz functions:

\[
B_{u} = L_{u} N_{u}, \tag{34}\]

\[
B_{\Phi} = L_{\Phi} N_{\Phi}. \tag{35}\]

Assembling all element contributions yields the overall system of equations which may be expressed as:

\[
\begin{bmatrix}
M_{uu} & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\ddot{u} \\
\ddot{\Phi}
\end{bmatrix}
+ 
\begin{bmatrix}
K_{uu} & K_{u\Phi} \\
K_{\Phi u} & -K_{\Phi\Phi}
\end{bmatrix}
\begin{bmatrix}
u \\
\Phi
\end{bmatrix}
= 
\begin{bmatrix}
f^{(e)}_{uu} \\
f^{(e)}_{\Phi}
\end{bmatrix}. \tag{36}\]

If we want to account for concentrated forces and also for concentrated electric charges, the external mechanical element force vector \(f^{(e)}_{uu}\) and the electric element charge vector \(f^{(e)}_{\Phi}\) have to be complemented with additional terms. The contribution of a concentrated point force at the jth-node of the system to the mechanical element force vector is given by:

\[
f^{(e)}_{uu,j} = \sum_{j=1}^{n_{u}} N_{u}^{T}(x_{j}) \tilde{F}_{j}, \tag{37}\]

and the contribution of a concentrated electrical charge to the element charge vector is equally expressed by:

\[
f^{(e)}_{\Phi,j} = \sum_{j=1}^{n_{\Phi}} N_{\Phi}^{T}(x_{j}) \tilde{Q}_{j}. \tag{38}\]

3. Numerical results

In order to illustrate the numerical performance of the algorithm, four benchmark examples are examined. The first benchmark problem is a piezoelectric bimorph beam, which is clamped at one end and free at the other. Thereafter, a carbon fiber reinforced plastic (CFRP) plate augmented with piezoelectric layers is considered. This example is used demonstrate active shape control of the plate loaded with a constant pressure distribution. Furthermore, to illustrate the behavior of the proposed finite element in time dependent processes two dynamic analyses are conducted. The eigenfrequencies of a piezoelectric disc are calculated and compared with an analytical approach and finally, a wave propagation problem is scrutinized. There Lamb waves are excited by surface-bonded piezoelectric patch actuators.
All results are compared to solutions obtained using either commercially available finite element software packages (ABAQUS/ANSYS) or results published in literature. To resolve boundary layers and singularities of the exact solution and to obtain exponential convergence a refined mesh is needed in those areas. A proper mesh therefore consists of a thin layer of elements (of the order of the thickness) around the boundaries and a geometrically refinement towards singularities such as re-entrant corners or points where concentrated loads are applied (Szabó and Babuška, 1991). For the proposed examples such a discretization strategy was adapted when possible. Furthermore, special shape functions including the singular behavior could be used to circumvent the aforementioned problem. Approaches similar to X-FEM and crack-tip finite elements could be useful (Belytschko and Black, 1999). But since this is not the objective of the paper the authors refrain from using these methods. Note that the contour plots in the following section are shown on a visualization mesh that does not conform to the finite element grid. The contour lines are only there to facilitate a better pot-processing.

3.1 Piezoelectric bimorph beam

The first benchmark example is a cantilever beam with piezoelectric material properties. This model was originally introduced by Tzou (1993) and later Gabbert et al. (1998) proposed this model as a suitable benchmark problem for piezoelectric structures. The beam consists of two layers of PVDF (polyvinylidene fluoride) with opposed polarization directions. The geometry and the boundary conditions are to be seen in Figure 3. The material properties of a single layer of the bimorph beam are given in Table I. The structure is discretized with 10 elements per layer ($n_x = 10$, $n_y = 1$, $n_z = 1$) utilizing different polynomial degree templates. An electric potential difference of 1 V between top and bottom surface is applied ($\pm 0.5$ V at top and bottom surface, respectively). This load results in a constant electric field through the thickness of the beam inducing a bending moment in the structure.

![Figure 3. Geometry and boundary conditions of the piezoelectric bimorph beam](image)

**Note:** All dimensions are given in (mm)

<table>
<thead>
<tr>
<th></th>
<th>$\nu$</th>
<th>$\varepsilon_{31}$</th>
<th>$\varepsilon_{32}$</th>
<th>$\varepsilon_{33}$</th>
</tr>
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<td>E</td>
<td>2 $\times 10^9$ N/m$^2$</td>
<td>0.29</td>
<td>$-0.046$ C/m$^2$</td>
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</table>

**Source:** Gabbert et al. (1998)
due to the difference in polarization. The results from the simulation with the presented finite elements are shown in Figure 4.

To verify the numerical results analytically the beam theory of Euler and Bernoulli is deployed. According to Marinković (2007) the bending moment generated by the electrical potential difference is given as:

$$ M = \frac{bh^2}{4} e_{31} E, $$

and the displacement field is determined by:

$$ u_z(x) = \frac{3}{2} \frac{e_{31} \Delta \Phi}{Eh^2} x^2. $$

As expected, the quality of the solution improves both with an increasing number of elements (h-refinement) and when utilizing higher order polynomials (p-refinement). It is well-known that the latter approach yields a faster convergence and hence is numerically more effective (Figure 5) (Szabo and Babuška, 1991). Figure 5 shows a convergence study comparing the rate of convergence if the h-version or the p-version of FEM is deployed.

The h-version model utilizes the well-known eight-node hexahedral elements. The mesh is generated such that the aspect ratio (ratio of smallest element dimension to largest) never exceeds the value of 1.4. Thus, all elements are almost cubically-shaped and no additional error is introduced through the approximation of the geometry. On the other hand the
p-version model uses only two elements, each describing one layer of the bimorph beam. Throughout the study the mesh is not altered, but the polynomial degree is varied from $p_x = p_y = p_z = 1$ to $p_x = p_y = p_z = 14$. The results are shown in Figure 5 and demonstrate the superior rate of convergence of the p-FEM.

The numerical solution using three-dimensional continuum elements will never fully coincide with the analytically predicted results, since three-dimensional elements are physically not restricted to the assumptions on which the beam theory is founded. Nonetheless, this example highlights that even slender structures can be simulated using higher order finite element approaches. This is an important result with respect to discretizing complex models. If only one single element type can be deployed to compute complex structures all problems arising with the coupling of different element types can be omitted (see constraint equations, Mortar Method, Arlequin method (Barthel and Gabbert, 2008) for instance).

The relative error between the analytical and numerical approach is given in Tables II and III for different models utilizing on the one hand an isotropic ansatz space and on the other hand an anisotropic one. A comparison between the proposed finite element and values available in literature is summarized in Table IV.

Deploying an anisotropic ansatz space is promising when dealing with thin-walled structures or in general with systems where one or two dimensions are considerably

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<th>dof = 3,472</th>
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<td>$(p_x = p_y = p_z = 1)$</td>
<td>$(p_x = p_y = p_z = 2)$</td>
<td>$(p_x = p_y = p_z = 3)$</td>
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<td>Error (%)</td>
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<table>
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<td>$(p_x = p_y = 2, p_z = 3)$</td>
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<td>Error (%)</td>
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<th>0.06</th>
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</tr>
<tr>
<td>Sze et al. (2004)</td>
<td>0.138</td>
<td>0.522</td>
<td>1.242</td>
<td>2.208</td>
<td>3.450</td>
</tr>
<tr>
<td>ABAQUS</td>
<td>0.150</td>
<td>0.586</td>
<td>1.294</td>
<td>2.277</td>
<td>3.534</td>
</tr>
<tr>
<td>Analytical</td>
<td>0.138</td>
<td>0.522</td>
<td>1.242</td>
<td>2.208</td>
<td>3.450</td>
</tr>
<tr>
<td>p-version</td>
<td>0.146</td>
<td>0.566</td>
<td>1.259</td>
<td>2.229</td>
<td>3.468</td>
</tr>
</tbody>
</table>

Table II. Relative error of the displacement at the tip of the beam ($x_1 = 0.1m$) for an isotropic distribution of the polynomial degree for various numbers of degrees of freedom (dof)

Table III. Relative error of the displacement at the tip of the beam ($x_1 = 0.1m$) for an anisotropic distribution of the polynomial degree for various numbers of degrees of freedom (dof)

Table IV. Static deflection ($u_z$) of the piezoelectric bimorph beam ($10^{-7}$ m)
smaller than the others. If Tables II and III are compared we note that solutions reaching the same quality can be obtained using fewer degrees of freedom in thickness direction when an anisotropic ansatz space is utilized.

Even if we use a coarse mesh with only a few degrees of freedom (dof = 1,260, Table II) acceptable results are obtained (Table IV). The quality of the solution is within an accuracy range which is sufficient for engineering applications.

3.2 Shape control of an active CFRP plate

The second static benchmark problem consists of a CFRP plate with a piezoelectric top and bottom layer. The active parts of the structure are used to suppress the deformation under a mechanical loading. The simulation model is shown in Figure 6.

The top as well as the bottom layer of the smart structure consist of the piezoelectric ceramic PZT G1195. The layers in between are made of a CFRP T300/976. The single plies are unidirectional with different fiber angles. The ply lay-up is $[0^\circ 90^\circ 0^\circ]_s$. The plate is simply supported in the middle plane of the plate in order to facilitate the comparison to results taken from literature. There the Kirchhoff plate theory has been applied. The upper surface is loaded with a constant pressure $p(x, y, z = h) = 200 \, \text{N/m}^2$.

The proposed hexahedral finite elements are used to solve this problem. In order to keep the numerical costs to a minimum an anisotropic distribution of the polynomial degree is considered. The polynomial degree in out-of-plane direction is limited $p_z = 1$. This limitation takes the physical dimensions of the plate into account. This approach is justified by the fact that the in-plane dimensions are drastically larger than the thickness. This constraint however does not corrupt the obtained results as the layer thickness is two orders of magnitude smaller than the in-plane dimensions. Using linear shape functions in thickness direction is similar to the classical laminate theory (CLT) which is viable for slenderness ratios comparable to this benchmark problem.

Each layer is meshed using one element over the thickness and nine elements within the plane of the plate ($n_x = 3, n_y = 3, n_z = 1$) utilizing an anisotropic polynomial degree template ($p_x = p_y = 6, p_z = 6, p_z = p_x = 1$). Thus, the total number of degrees of freedom amounts to dof = 11,191. Three simulation are executed applying different values of a constant electric potential ($\Delta \Phi = 0 \, \text{V}, \Delta \Phi = 15 \, \text{V}, \Delta \Phi = 27 \, \text{V}$) between top and bottom layer. Those results depict the evolution of the displacement field and show

![Figure 6. Model of an active laminate to assess the shape control capabilities of piezoelectric ceramics](image_url)

**Notes:** The dimensions of the studied plate are: $a = 0.254 \, \text{m}$; $b = 0.254 \, \text{m}$; $h = 0.001336 \, \text{m}$
how the mechanical deformation can be reduced utilizing active structures. The material properties of the single layers are listed in Table V.

Figure 7(a)-(c) show surface plots of the displacement field with respect to the global $z$-direction. The ability of the increasing electric field to counteract the effect of the mechanical pressure is well documented in Figures 7 and 8. The maximum deformation in global $z$-direction is reduced from approximately $u_{z,0} = -0.26 \text{ mm}$ to $u_{z,27} = -0.013 \text{ mm}$. This means that only 5 percent of the initial maximum deformation remains when the electric field is adjusted accordingly. These results highlight the shape control properties of smart structures. Adaptive designs like this can be deployed when high precision is demanded and shape deviation interferes with the faultless mode of operation.

Again, the results calculated using the proposed elements are compared to a converged ABAQUS reference solution. The convergence was assessed by considering the material properties of the single layers

<table>
<thead>
<tr>
<th>Properties</th>
<th>PZT G1195</th>
<th>T300/976</th>
</tr>
</thead>
<tbody>
<tr>
<td>Layer thickness</td>
<td>0.254 mm</td>
<td>0.138 mm</td>
</tr>
<tr>
<td>$S_{11}^E$</td>
<td>$1.587 \times 10^{-11} \text{ m}^2/\text{N}$</td>
<td>$6.666 \times 10^{-12} \text{ m}^2/\text{N}$</td>
</tr>
<tr>
<td>$S_{22}^E$</td>
<td>$1.587 \times 10^{-11} \text{ m}^2/\text{N}$</td>
<td>$1.111 \times 10^{-10} \text{ m}^2/\text{N}$</td>
</tr>
<tr>
<td>$S_{44}^E$</td>
<td>$4.132 \times 10^{-11} \text{ m}^2/\text{N}$</td>
<td>$1.408 \times 10^{-10} \text{ m}^2/\text{N}$</td>
</tr>
<tr>
<td>$S_{66}^E$</td>
<td>$4.132 \times 10^{-11} \text{ m}^2/\text{N}$</td>
<td>$4.000 \times 10^{-10} \text{ m}^2/\text{N}$</td>
</tr>
<tr>
<td>$\nu_{12}$</td>
<td>0.3</td>
<td>0.018</td>
</tr>
<tr>
<td>$\nu_{23}$</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>$e_{31}$</td>
<td>$-22.86 \text{ C/m}^2$</td>
<td>$-22.86 \text{ C/m}^2$</td>
</tr>
<tr>
<td>$e_{32}$</td>
<td>$-22.86 \text{ C/m}^2$</td>
<td>$-$</td>
</tr>
</tbody>
</table>

Note: The piezoelectric layer is polarized in $z$-direction
Source: Kioua and Mirza (2000)

![Contour plot of the displacement field in global z-direction (u_z) for the active CFRP plate under different electric fields and a constant mechanical surface load (p(x, y, z = h) = 200 N/m^2)](image)
the strain energy as well as the maximum displacement in z-direction. Utilizing the same hexahedral elements (C3D20, C3D20E) as in the previous section 191823 degrees-of-freedom are needed to reach a comparable accuracy as the p-FEM simulation. The comparison is shown in Figure 8. A good agreement between the results is achieved. The error is less than 0.1 percent although only a twentieth of the number of degrees of freedom is used to compute the p-version solution.

3.3 Modal analysis of a piezoelectric circular disc

To quantify the dynamical properties of the p-version of the FEM the eigenfrequencies and mode shapes of a circular disc are to be determined. The disc consists of PIC-151 (material properties are given in Table VI). The diameter and thickness of the circular plate are \( d = 0.03 \) m and \( t = 0.001 \) m, respectively. Two different cases are studied. For both models no mechanical boundary conditions are applied. To illustrate the influence of the electro-mechanical coupling on the eigenfrequencies of the circular plate the electrodes are short-circuited for the first example and open for the second one. In the first case the coupling between electrical and mechanical properties vanishes (Piefort, 2001) and thus the results can be compared to an analytical solution of a free elastic circular plate (Giurgiutiu, 2008). In the second case a charge separation takes place resulting in an electric field. These coupling effects stiffen the material and hence one obtains higher values for the eigenfrequencies.

---

Figure 8. Displacement field in z-direction along the path \( y = b/2, z = h/2 \) for a variable x

<table>
<thead>
<tr>
<th>Mechanical properties</th>
<th>Piezoelectric properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_{E1} )</td>
<td>( 16.38 \times 10^{-12} ) m(^2)/N</td>
</tr>
<tr>
<td>( S_{E3} )</td>
<td>( 19.00 \times 10^{-12} ) m(^2)/N</td>
</tr>
<tr>
<td>( S_{E5} )</td>
<td>( 50.96 \times 10^{-12} ) m(^2)/N</td>
</tr>
<tr>
<td>( S_{E12} )</td>
<td>( -5.66 \times 10^{-12} ) m(^2)/N</td>
</tr>
<tr>
<td>( S_{E13} )</td>
<td>( -7.11 \times 10^{-12} ) m(^2)/N</td>
</tr>
<tr>
<td>( S_{E4} )</td>
<td>( 50.96 \times 10^{-12} ) m(^2)/N</td>
</tr>
<tr>
<td>( S_{E6} )</td>
<td>( 44.97 \times 10^{-12} ) m(^2)/N</td>
</tr>
</tbody>
</table>

Table VI. Material properties of PIC-151

| (\( \varepsilon_0 = 8.8542 \times 10^{-12} \) As/Vm) polarized in z-direction of the structure |
For the sake of clarity, we recall equation (36) given in Section 2.3 at this point. Hence, the equation of motion for piezoelectric structures can be expressed in terms of the FEM as:

$$
\begin{bmatrix}
M_{uu} & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\ddot{u} \\
\Phi
\end{bmatrix}
+ \begin{bmatrix}
K_{uu} & K_{u\Phi} \\
K_{\Phi u} & -K_{\Phi \Phi}
\end{bmatrix}
\begin{bmatrix}
u \\
\Phi
\end{bmatrix}
= \begin{bmatrix}
f_{uu} \\
f_{\Phi u}
\end{bmatrix}.
$$

If we want to conduct a modal analysis, two special cases of electric boundary conditions (short-circuited and open electrodes) are of interest. If the electrical potential is given, i.e. it is a priori known, the second row of equation (36) can be rearranged as:

$$
M_{uu} \ddot{u} + K_{uu} u = f_{uu} - K_{u\Phi} \Phi,
$$

where the second term on the right hand side of the equation describes the equivalent piezoelectric loads (Piefort, 2001). From this equation it can be inferred that the eigenvalue problem of a piezoelectric structure with controlled electric potential is equal to the case of short-circuited electrodes ($\Phi = 0$):

$$
(K_{uu} - \omega^2 M_{uu}) u = 0.
$$

Thus, the resonance frequencies ($\omega/2\pi$) and eigenmodes of this system are the same as if no electro-mechanical coupling would exist.

On the other hand, if we regard boundary conditions resembling open electrodes, equation (36) cannot be simplified as shown in the previous example. To simulate open electrodes a zero charge boundary condition ($f_{\Phi u} = 0$) has to be specified (Piefort, 2001). Hence, we obtain:

$$
\Phi = K_{\Phi \Phi}^{-1} K_{\Phi u} u.
$$

Substituting equation (43) into equation (36) yields:

$$
M_{uu} \ddot{u} + \left( K_{uu} + K_{u\Phi} K_{\Phi \Phi}^{-1} K_{\Phi u} \right) u = f_{uu}.
$$

Thus, the eigenvalue problem for open electrodes can be written as:

$$
(K^* - \omega^2 M_{uu}) u = 0,
$$

which clearly shows that the natural frequencies, as well as the mode shapes are now influenced by the electrical properties of the piezoelectric material. The electro-mechanical coupling increases the system stiffness if open electrodes are specified. Hence, the natural frequencies are increased compared to the purely elastic case.

The different results are summarized in Table VII. The results of the short-circuited electrodes case are presented in columns 3-5 and the results of the open electrodes case can be taken from columns 6 and 7, respectively. To verify the results obtained using the proposed finite element (models (3) and (5)) the commercial software package
### Table VII.
First eight non-zero transversal eigenfrequencies in kHz and contour plots of the mode shapes of a circular plate

<table>
<thead>
<tr>
<th>Number</th>
<th>Mode shape</th>
<th>Eigenfrequencies of the two models</th>
<th>Open electrodes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Short-circuited electrodes</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td><img src="image1" alt="Mode shape 1" /></td>
<td>3.13</td>
<td>3.114</td>
</tr>
<tr>
<td>2</td>
<td><img src="image2" alt="Mode shape 2" /></td>
<td>5.4</td>
<td>5.413</td>
</tr>
<tr>
<td>3</td>
<td><img src="image3" alt="Mode shape 3" /></td>
<td>7.3</td>
<td>7.215</td>
</tr>
<tr>
<td>4</td>
<td><img src="image4" alt="Mode shape 4" /></td>
<td>12.23</td>
<td>12.084</td>
</tr>
<tr>
<td>5</td>
<td><img src="image5" alt="Mode shape 5" /></td>
<td>12.84</td>
<td>12.534</td>
</tr>
<tr>
<td>6</td>
<td><img src="image6" alt="Mode shape 6" /></td>
<td>19.7</td>
<td>19.043</td>
</tr>
<tr>
<td>7</td>
<td><img src="image7" alt="Mode shape 7" /></td>
<td>21.02</td>
<td>20.454</td>
</tr>
<tr>
<td>8</td>
<td><img src="image8" alt="Mode shape 8" /></td>
<td>22.97</td>
<td>22.386</td>
</tr>
</tbody>
</table>

**Notes:** Short-circuited electrodes (1 – analytical solution, 2 – ABAQUS/C3D20, 3 – hexahedral finite element based on the p-version FEM); open electrodes (4 – ABAQUS/C3D20E, 5 – hexahedral finite element with piezoelectric degrees of freedom based on the p-version FEM)
ANSYS (models (2) and (4)) is deployed, and for the purely elastic case an analytical solution (model (1)) based on the plate theory exists.

The ANSYS reference solution deploys fully integrated hexahedral elements with quadratic shape functions (Solid226). In order to reach a solution with a relative error of approximately 2.5 percent for the eighth eigenfrequency the circular plate is discretized with 816 elements amounting to 23,820 degrees of freedom. To obtain a slightly better accuracy (2.3 percent) as the reference solution the simulation utilizing the p-version finite element is discretized with 51 elements (swept mesh) using a polynomial degree template of $p_x = p_r = 4, p_\eta = p_\phi = 4$ and $p_z = 3$ (dof = 13,712). Both methods show good agreement with the analytical solution for the purely elastic case. The computed eigenfrequencies for the piezoelectric continuum almost coincide. If we take a closer look at Table VII we notice that not all eigenfrequencies are similarly influenced. The second, fifth, seventh and eighth eigenfrequencies are more affected by the electrical-mechanical coupling. Studying the eigenforms of the mentioned eigenfrequencies leads to the assumption that primarily bending-dominant modes are influenced by the coupling of mechanical and electrical field variables. Both the p-version finite element as well as the standard quadratic hexahedral finite element predicts the same behavior. Generally, the short-circuited case has lower eigenfrequencies as the case with open electrodes. In the previous paragraph the reason has been theoretically explained in detail with help of equations (42) and (45). Similar results have been obtained deploying isogeometric finite elements (Willberg and Gabbert, 2012).

3.4 Lamb wave propagation problem
As a further example of the capabilities of the proposed element ultrasonic wave propagation in a plate is chosen. As described first by Horace Lamb in his article “On Waves in an Elastic Plate” (Lamb, 1917) and accordingly named after him, Lamb waves arise when a thin-walled structure is excited. Despite their dispersive behavior and the fact that for every given frequency at least two modes exist (symmetric and anti-symmetric), Lamb waves are still frequently used for SHM purposes (Boller et al., 2009). This is due to their favorable properties like long-range inspection capabilities and high sensitivity towards structural damages. These features provide a good foundation for designing online monitoring devices. Since a future focus will be on studying wave propagation with respect to such SHM applications the feasibility of the p-version finite element with respect to capturing ultrasonic guided wave propagation is analyzed. Figure 9 contains the model to evaluate the Lamb wave propagation as well as schematically displays the methodology to extract the needed results. The wave propagation is excited using two collocated piezoelectric patch actuators as shows in Figure 9(a). These are assumed to be perfectly bonded to the host structure. That is to say the effects introduced by the adhesive layer like the shear lag effect can be neglected. The extraction of the time history of the displacement field at Point A and B, respectively, is shown in Figure 9(b). The distances between the actuators and points of measurement can be taken arbitrarily and do not influence the quality of the gathered displacement signals. The diagonal length of the plate $l_p$ should be chosen such that no reflections from the boundary pollute the solution at points A and B. Thus, $l_p$ depends both on the excitation frequency and the thickness of the regarded plate which determine the group speed the Lamb wave packet is travelling with.
To determine the quality of the finite element solution the group speed of the propagating Lamb wave packet is utilized. The group speed computed using the p-version of the Finite-Element-Method is compared to the value given by the analytical solution (Vivar-Perez, 2012). In order to extract the group velocity from the finite element solution the time history signal of the displacements at the regarded point is subjected to a Hilbert transform:

\[
H_{A,B}(u(t)) = \frac{1}{\pi} \int_{-\infty}^{\infty} u_{A,B}(\tau) \cdot \frac{1}{t-\tau} \, d\tau
\]  (46)

and the time of flight \(t_c\) between two points A and B is measured (Figure 9). The time of flight is evaluated by comparing the position of the centroid of the envelope of the time signal and its centroid.

**Notes:**
(a) Cross section of the plate along its diagonal with definition of the points A and B to estimate the time of flight; \(\psi\) marks the diagonal of the plate, \(\omega\) is the circular excitation frequency and \(n\) describes the number of cycles in the excitation signal; \(l_a, l_b, l_p\) are the distance between the center of the plate and plate A, point B and the corner of the plate, \(d\) denotes the thickness of the investigated plate; \(a = \pm 1\) depending on whether the \(A_0\) or \(S_0\) mode ought to be excited; note that the applied forces should only envision the effect exercised by the piezoelectric patch actuators;
(b) evaluation of time of flight \(t_c\) using Hilbert transform to compute the envelope of the time signal and its centroid.

Figure 9. Numerical model and schematic representation of the method used to evaluate the results of the conducted wave propagation analysis.
history signal at those two points. The envelope can be computed by the following relation:

\[ e(t) = \sqrt{H_{A,B}(u_{A,B}(t))^2 + u_{A,B}(t)^2} \]  

and its centroid is gained computing the statical moment of the envelope:

\[ t_{A,B} \frac{\int_{t_{end}}^{t_{end}} e_{A,B}(t) \cdot t \, dt}{\int_{t_{0}}^{t_{end}} e_{A,B}(t) \, dt} \]  

(48)

Since the distance between point A and B is known, the group velocity \( c_g \) of the Lamb wave modes can be estimated as:

\[ c_g = \frac{l_b - l_a}{t_B - t_A} \]  

(49)

Exemplarily, the results using an isotropic aluminum plate (\( E = 70 \text{ GPa}, v = 0.33 \)) with a surface-bonded piezoelectric actuator (PIC-151 see Table VI) are shown in Figure 10. Due to the symmetry of the structure only one quarter of the plate has to be modeled. The discretized structure is shown in Figure 11. The mesh consists of 793 hexahedral elements.

Notes: The dimensions of the plate are 0.2 m × 0.2 m × 0.002 m; the piezoelectric actuators dimensions are 0.005 m × 0.005 m × 0.001 m.

Figure 10. Model of the aluminum plate deployed to study Lamb wave propagation

Figure 11. Lamb wave propagation with a clear separation of \( A_0 \) and \( S_0 \)-mode
elements with mechanical capabilities and of 9 hexahedral elements with piezoelectric properties \( n_x = 25, n_y = 25, n_z = 1, \text{dof} = 69,312 \). To account for the thin-walled structure an anisotropic polynomial degree template is utilized \( (p_x = 3, p_y = 3, p_z = 4) \). The propagation of the wave as well as the dispersive nature of Lamb waves can be clearly seen in Figure 11. Further, a clear separation of the faster symmetric Lamb wave mode \( (S_0) \) from the anti-symmetric one \( (A_0) \) is to be noticed. The shear horizontal mode \( (SH_0) \) is also excited but cannot be seen in this figure due to the small amplitudes of the in-plane displacement components. It has to be noted that the out-of-plane displacement component of the \( SH_0 \)-mode is zero. A simulation run with the proposed discretization captures the behavior of the wave quite well. With respect to the group velocity a relative error compared to an analytic solution (Vivar-Perez, 2012) of only 5.6 percent for the anti-symmetric wave propagation and 0.1 percent for the symmetric one is accomplished.

Additional convergence studies have shown that a numerically cost-effective simulation can be executed utilizing an in-plane polynomial degree of \( 3 (p_x = p_y = 3) \), and an out-of-plane polynomial degree of \( 4 (p_z = 4) \). It is to say that this polynomial degree template is optimal in the sense that it is numerically efficient (Duczek et al., 2012). The best ratio between computational costs and quality of the solution is achieved using this anisotropic ansatz space. The results of the convergence study conducted can give an estimate on the degrees of freedom per wavelength needed in order to ensure results within an acceptable range of the analytically predicted value. Figure 12 shows a graph of the degrees of freedom per wavelength versus the relative error in the group velocity, confirming that an anisotropic polynomial degree template is best suited to compute ultrasonic guided wave propagation when deploying higher order finite elements based on the normalized integrals of the Legendre polynomials. Thus, it is possible to estimate the quality of the solution a priori or to generate a cost-effective discretization and consequently safe computational time. In order to obtain the best results in as little time as possible such convergence studies are rather helpful. The author’s current research has shown that different higher order approaches like the spectral FEM or \( C_0 \)-continuous isogeometric analysis yield the same polynomial degree template (Duczek et al., 2012).

In order to check if the amplitudes of the displacement field can be captured as well as the group velocity of the travelling Lamb wave packet, Figure 13 shows the magnitude of the displacement at point A over time:

\[
  u^A_{\text{mag}}(t) = \sqrt{u^A_x(t)^2 + u^A_y(t)^2 + u^A_z(t)^2}.
\]
In Figure 13 a good agreement between the analytical and the numerical solution can be observed. Both curves are virtually coincident highlighting the capabilities of the p-version of the FEM. For comparably accurate results conventional low order finite element approaches need well over half a million degrees of freedom.

4. Conclusion and outlook
This article presents the development of a new piezoelectric finite element based on the p-version of the Finite-Element-Method (at least to the author’s knowledge no piezoelectric finite element based on the Hierarchical-Finite-Element-Method has been published in literature, yet). The development of higher order coupled-field finite elements holds several advantages if smart structures are considered. Due to their inherent property to utilize anisotropic ansatz spaces complex designs can be readily simulated. The current paper presented several static as well as dynamic benchmark tests to demonstrate the capabilities of the proposed element. All numerical examples have been found to agree well with previously published results or with numerical reference solutions. It has been shown that good results can be obtained with far less degrees of freedom. Thus, the proposed finite element can lead to a significant reduction in the overall numerical costs.

For future applications SHM using a network of piezoelectric, surface-bonded actuators and sensors is of primary interest. In thin-walled structures the applied actuators excite Lamb waves to measure the state of “health” of the system under consideration. Such SHM-systems promise a vast potential for saving maintenances (from scheduled maintenance intervals to on-demand maintenance) costs and to improve the safety as well as the utilization of material. Thus, the proposed finite element is a step towards a holistic numerical treatment of SHM-related problems using p-version finite elements.

References


Further reading


About the authors

Sascha Duczek studied Mechanical Engineering at the University of Magdeburg, Germany and received his Diploma in 2010. Since then he has been working at the University of Magdeburg on his PhD thesis. His research interests include numerical mechanics and structural health monitoring. Sascha Duczek is the corresponding author and can be contacted at: sascha.duczek@st.ovgu.de

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