Development, Validation and Comparison of Higher Order Finite Element Approaches to Compute the Propagation of Lamb Waves Efficiently

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Abstract. When considering structural health monitoring (SHM) applications efficient and powerful numerical methods are required to predict the behavior of ultrasonic guided waves and to design SHM systems. The existing commercial explicit finite element analysis tools based on standard linear displacement elements quickly reach their limits when applied to ultrasonic waves in thin plates, so called Lamb waves. It is known that the required temporal and spatial resolution causes enormous computational costs. One resort to overcome this problem is the application of special finite elements utilizing higher order polynomial shape functions ($p > 2$). The current paper is focused on the development of such higher order finite elements and the verification of their accuracy and performance. In the paper we compare and evaluate the capabilities of spectral finite elements, $p$-version finite elements and isogeometric finite elements. Their advantages and disadvantages with respect to ultrasonic wave propagation problems are discussed and their properties are demonstrated by solving a benchmark problem. Higher order finite elements with varying polynomial degrees in longitudinal and transversal direction (anisotropic ansatz space) are investigated and convergence studies are performed. The results of the convergence studies are summarized and a guideline to estimate the optimal discretization is prepared. If the required accuracy is known, the proposed guideline provides a helpful means to determine both the element size and the polynomial degree template for a given model.

Introduction

In 1917 a special type of elastic guided waves, propagating in thin-walled structures with free boundaries, has been described by Horace Lamb [1]. Consequently, this type of guided waves has been named “Lamb” waves. Nowadays they are quite common for structural health monitoring applications [2]. Considering plates made of homogenous and isotropic material two basic types of Lamb waves are to be distinguished [3], namely symmetric and antisymmetric modes. Characteristic features of these ultrasonic guided waves are their dispersive behaviour and the existence of at least two modes at every frequency [4]. Under certain conditions a conversion of the symmetric mode into the antisymmetric one or vice versa can be observed. Mode conversions occur when the propagating wave packet encounters damage within the considered structure or at certain design elements, like holes or stringers. Despite the mentioned complex propagation properties there are important features making Lamb waves very interesting for SHM purposes. One characteristic feature is their small wavelengths in higher frequency ranges making them sensitive towards structural defects.

From a numerical point of view the simulation of ultrasonic guided waves is a highly demanding task. Using Finite-Element-Methods (FEM) in the time domain requires a rather fine temporal as well as spatial discretization to resolve the short wavelengths needed for damage detection and the high frequency range at which SHM systems are operated. Since conventional linear pure displacement finite elements quickly reach their limit when dealing with ultrasonic guided waves, alternative discretization approaches are proposed. The focus of the paper is on the following higher order FEM-schemes: (i) the Hierarchical $p$-FEM based on the normalized integrals of the Legendre-
polynomials [5, 6], (ii) the Spectral Element Method (SEM) utilizing Lagrange-polynomials on a Gauss-Lobatto-Legendre grid [7, 8] and (iii) the N-FEM using Non-Uniform-Rational-B-Splines (NURBS) as shape functions [9, 10]. So far, only SEM has been deployed to high frequency wave propagation problems and the latter two approaches have been primarily utilized for static problems including non-linear analyses, plasticity, etc. [11–13]. The comparison of these numerical schemes is focused on finding optimal polynomial degree distributions, minimizing the computational costs and memory storage requirements. The article is divided into three main parts. The first part provides a brief summary of the basic principles of the Finite-Element-Method and an overview of the analyzed shape function types, namely the normalized integrals of the Legendre polynomials, the Lagrange polynomials and the NURBS. The focus of the second part is on presenting the convergence studies and the implications for deriving an optimal discretization scheme. Finally, part number three contains conclusions and future research proposition.

Finite-Element-Method

Our starting point is the well-known linear system of equations for finite element applications, describing the undamped motion of a body

\[ M \ddot{U} + KU = \mathbf{f} \]  

(1)

with the mass matrix \( M \), the stiffness matrix \( K \) and the load vector of the external forces \( \mathbf{f} \). A detailed derivation of Eq. 1 can be found in standard text books on FEM [14, 15].

Higher order shape functions

In this section three types of shape functions are introduced which can be used to generate higher order finite element schemes. The first type being presented, are Lagrange polynomials which are very popular for SEM approaches. Thereafter, the normalized integrals of the Legendre polynomials are described. They are the foundation of the p-version of the Finite-Element-Method. Finally, non-uniform rational B-splines are presented which form the basis of the so called isogeometric or N-FEM.

Lagrange polynomials. In general, Lagrange interpolation polynomials are defined by

\[
N^{1D, \text{Lagrange}, p}_n(\xi) = \prod_{j=1, j\neq a}^{p+1} \frac{\xi - \xi_j}{\xi_n - \xi_j}, \quad n = 1, 2, \ldots, (p + 1).
\]

(2)

In [16] it has been shown that the roots of the Lobatto polynomials are an appropriate choice for SEM purposes. Hence, shape functions of order \( p \) are based on the set of nodes \( \xi_j \) consisting of the roots \( \xi_{0,a}^{L0,p-1} \) of the \((p-1)\)-order Lobatto polynomial with \( a = 1, \ldots, (p-1) \) including the interval boundaries \( \pm 1 \)

\[
\xi_n = \begin{cases} 
  n = 1 & \rightarrow -1 \\
  2 \leq n < p + 1 & \rightarrow \xi_{0,a}^{L0,p-1} \\
  n = p + 1 & \rightarrow +1 
\end{cases}
\]

(3)

Legendre polynomials. Legendre polynomials are numerically determined by Bonnet’s formula

\[
L_n(\xi) = \frac{1}{n + 1} [(2n + 1)\xi L_n(\xi) - n L_{n-1}(\xi)], \quad n = 1, 2, \ldots, -1 \leq \xi \leq 1
\]

(4)

with \( n \) denoting the polynomial degree. The recursion starts with the first two Legendre polynomials
In order to construct shape functions for p-elements the normalized integrals of the Legendre polynomials \( \Phi_n(\xi) \) are used

\[
\Phi_n(\xi) = \frac{2n-1}{2} \int_{-1}^{1} L_{n-1}(z) d\xi = \frac{1}{\sqrt{4n-2}} [L_n(\xi) - L_{n-1}(\xi)], \quad n = 2,3,\ldots
\]

Thus, the value of the modified Legendre polynomials is equal to zero at the boundaries \( \xi = \pm 1 \). The linear Lagrange polynomials constitute the first two shape functions

\[
N_{1}^{1D,\text{Legendre}}(\xi) = \frac{1-\xi}{2}, \quad (7a)
\]

\[
N_{2}^{1D,\text{Legendre}}(\xi) = \frac{1+\xi}{2}. \quad (7b)
\]

Higher order ansatz functions \( N_{n}^{1D,\text{Legendre}} \) are generated using Eq. (6) in the following way

\[
N_{n}^{1D,\text{Legendre}}(\xi) = \Phi_{n+1}(\xi), \quad n = 3,4,\ldots
\]

**NURBS.** In the following paragraph, non-uniform rational B-splines are introduced. The first order basis functions \( R_{i,0}(\beta) \) of the polynomial degree \( p = 0 \) are

\[
R_{i,0}(\beta) = \begin{cases} 1, & \text{if } \beta \in [\beta_i, \beta_{i+1}) \\ 0, & \text{otherwise} \end{cases}
\]

The basis functions \( R_{i,p}(\beta) \) of higher orders \( p > 0 \) are defined as

\[
R_{i,p}(\beta) = \frac{\beta - \beta_{i,p-1}}{\beta_{i+p} - \beta_{i}} R_{i,p-1}(\beta) + \frac{\beta_{i+p+1} - \beta}{\beta_{i+p+1} - \beta_{i+1}} R_{i+1,p-1}(\beta).
\]

The indices \( i \) and \( p \) denote the \( i^{th} \) basis function of polynomial order \( p \). Utilizing B-spline basis functions \( R_{i,p}(\beta) \), NURBS basis functions can be computed using

\[
N_{i}^{1D,\text{NURBS},p}(\beta) = \frac{R_{i,p}(\beta) w_i}{\sum_{j=1}^{n_{\text{nodes}}} R_{j,p}(\beta) w_j}, \quad i = 1,2,\ldots,(p+1).
\]

For further details concerning the derivation of NURBS shape functions it is referred to [17, 18].

**Comparison of the different types of higher order shape functions.** In the following paragraph the properties of the one-dimensional shape functions are summarized and compared to each other to highlight the advantages and disadvantages of using one or the other set of shape functions. Tab. 1 summarizes important properties pertaining to higher order shape functions.

<table>
<thead>
<tr>
<th>Property</th>
<th>Lagrange</th>
<th>Legendre</th>
<th>NURBS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inter element continuity</td>
<td>( C^0 )</td>
<td>( C^0 )</td>
<td>( C^{p-1} ) or less</td>
</tr>
<tr>
<td>Common dof of adjacent elements</td>
<td>1</td>
<td>1</td>
<td>( p ) or less</td>
</tr>
<tr>
<td>Reference domain</td>
<td>( W = [-1, 1] )</td>
<td>( W = [-1, 1] )</td>
<td>( W = [0, 0.5] )</td>
</tr>
<tr>
<td>Physical interpretation of dof</td>
<td>yes</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>Hierarchic basis</td>
<td>no</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>Function values can be negative</td>
<td>yes</td>
<td>yes</td>
<td>no</td>
</tr>
</tbody>
</table>
First of all the inter element continuity has to be mentioned. An advantage of isogeometric elements is that the continuity can be varied between $C^{p-1}$ and $C^0$, whereas spectral elements and $p$-elements are normally formulated in a way to fulfill only the minimal continuity requirements posed by the variational formulation. Therefore, they exhibit mainly $C^0$ continuity. On the other hand Lagrange polynomials hold the advantage that all degrees-of-freedom correspond to physical nodes and consequently retain a physical meaning. This makes the pre- and post-processing easier and an additional processing of the simulation results is not necessary. The normalized integrals of the Legendre polynomials are constructed using a hierarchical basis. That is to say, shape functions of lower polynomial degrees are also present in higher order approaches. This is an attractive feature of this set of shape functions, since the hierarchy of the basis functions has also an immediate consequence on the structure of the resulting system matrices. All system matrices corresponding to the polynomial orders 1 to $p - 1$ are sub-matrices of the system matrix corresponding to polynomial degree $p$. The one-dimensional shape functions for a polynomial degree of $p = 4$ are exemplarily shown in Fig. 1.

**Two-dimensional higher order shape functions.** The two-dimensional shape functions are simply formed by building the tensor product of the one-dimensional ones

$$N_{2D,\text{type}}^k(\xi, \eta) = N_{1D,\text{type}}^m(\xi)N_{1D,\text{type}}^n(\eta),$$

(12)

with $k = 1, 2, \ldots, L$, $m = 1, 2, \ldots, (p_\xi + 1)$ and $n = 1, 2, \ldots, (p_\eta + 1)$. $L$ denotes the total number of degrees-of-freedom for one finite element ($L = (p_\xi + 1) (p_\eta + 1)$). The development of the proposed higher order finite elements is based on utilizing the tensor product space [5] in contrast to the frequently applied trunk space.

**Convergence studies**

This section presents the numerical model used to evaluate the convergence properties of the three proposed shape functions sets as well as the methodology to extract meaningful results from the simulations.

**Definition of the mathematical model.** The starting point for the convergence studies is a two-dimensional plane strain model. The geometrical set up is depicted in Fig. 2. The length of the plate is chosen such that no reflections from the right boundary are affecting the signals at the two measuring points during the simulation time. In our model the excitation is accomplished using collocated point forces in global $x_2$-direction.
As excitation signal a sine-burst is used (Eq. 13).

Their time-dependent amplitudes follow a sine function modulated utilizing a Hann-window

\[
F(t) = \begin{cases} 
\hat{F} \sin(2\pi f_D t) \sin^2(t \cdot f_D / n_D), & \text{if } 0 \leq t \leq n_D / f_D, \\
0, & \text{if } t > n_D / f_D.
\end{cases}
\]  

(13)

Here \(f_D\) denotes the central excitation frequency. Using collocated concentrated forces allows us to benefit from a mono-modal excitation. Since an excitation frequency of \(f_D = 477.5\) kHz and a plate made of aluminum with a thickness of 2 mm are chosen only a single mode (A0) is present in the numerical model (Fig. 2).

For the remainder of this article it is assumed that only the first antisymmetric mode is excited. Due to its smaller wavelengths it is sufficient to study the behaviour of the A0-mode only. If the discretization is fine enough to resolve the antisymmetric mode, the symmetric one is resolved as well.

**Procedure to evaluate the simulation results.** An appropriate indicator to assess the quality of the numerical solutions the is time of flight \((t_c)\). To extract the time of flight the Hilbert-transform is applied to the time history of the displacements saved at points A and B. Utilizing the Hilbert-transform an envelope of the time signal can be determined and the centroid of this envelop as well. \(t_c\) is then computed using the coordinates of the centroid [16]

\[
t_c = t_B - t_A.
\]  

(14)

**Discretization.** In order to isolate the influence of the discretization in thickness \(x_2\) and axial direction \(x_1\) on the quality of the results, two convergence studies have been performed. At first the discretization in global \(x_1\)-direction is investigated. To avoid any influence of the mesh density in \(x_2\)-direction of the plate the structure is discretized utilizing one finite element with a fixed polynomial degree of \(p_{x2} = 4\). This polynomial degree guarantees that the measured signal quality is only influenced by \(p_{x1}\). Then an \(h\)-refinement for the spectral and \(p\)-elements and a \(k\)-refinement [19] for the isogeometric elements are conducted for various polynomial degrees \(p_{x1} = 2, 3 \ldots, 6\). Thereafter, a fixed mesh in the longitudinal direction is chosen to evaluate the effects of the discretization in \(x_2\)-direction. The mesh is constructed in a way such that it does not interfere with intended purpose. More than 20 pseudo-nodes per wavelength utilizing finite elements with a longitudinal polynomial degree of \(p_{x1} = 6\) are deployed to ensure an unaffected evaluation. The convergence study is then performed varying \(p_{x2}\) between 1 and 8 (\(p\)-refinement). After determining the optimal discretization parameters of the three presented finite element approaches these methods are compared with standard linear elements, utilizing both fully and selectively reduced integrated elements.
Results of the convergence studies

This section provides the fundamental results in order to derive guidelines on how to choose the polynomial degree and/or the mesh density for a given problem in order to obtain good quality results using as little numerical effort as possible.

Discretization in longitudinal direction. Fig. 3a to 3e display the results of the convergence with respect to a varying polynomial degree in global $x_1$-direction. These figures display pseudo-nodes per wavelength ($\chi$) versus relative error ($E_{rel}$) in the time of flight curves. For the case of spectral and conventional lower order finite elements the so called pseudo-nodes correspond to physical nodes. When dealing with N-FEM or p-FEM they do not represent geometrical nodes and serve only as a distinct measure of the mesh density. For a given model the value of $\chi$ is given by

$$\chi_{k_0} = \frac{\text{dof}}{2(p_{x_2} + 1)^{l_p}} \lambda_{k_0}.$$  

(15)

$\lambda_{A0}$ denotes the wavelength of the antisymmetric Lamb wave mode. The relative error with respect to the analytical solution is computes utilizing the numerically ($t_{c,\text{num,type}}$) and analytically ($t_{c,\text{ana}}$) obtained time of flights

$$E_{rel} = \left[\frac{t_{c,\text{num,type}} - t_{c,\text{ana}}}{t_{c,\text{ana}}}\right] \cdot 100\%.$$

(16)

As expected, the convergence rate is the faster, the higher polynomial orders are. The curves are steady except for small peaks experienced in the convergence curves of spectral (Lagrange) and $p$-version (Legendre) finite elements. Local element eigenfrequencies are causing these discontinuities as conclusively explained in [16]. The same behaviour is to be observed if $C^{0\text{I}}$-continuous isogeometric finite elements are used. One point which can be taken from Fig. 3a to 3e is that a low number of pseudo-nodes per wavelength ($\chi < 10$) is sufficient to approximate the $A_{0\text{I}}$-mode accurately, taken a threshold for the relative error of 1% into account. As mentioned before the $S_{0\text{I}}$-mode is not considered as the difference in wavelengths entails that if the $A_{0\text{I}}$-mode is computed accurately the results for the symmetric mode are also of high quality. When analyzing the results of the proposed numerical approaches we note that the finite elements based on the normalized integrals of the Legendre polynomials offer the lowest relative error for $p_{x_1} = 5, 6$. Regarding the remaining results the convergence rate of $p$-elements and spectral elements is the same and their curves are virtually coincident. Furthermore, we observe that isogeometric finite elements exhibit the fastest convergence rates and thus, need the lowest amounts of pseudo-nodes per wavelength to achieve a relative error threshold of 1 %. Consequently, they require the least memory to store the system matrices. Further studies have shown that this effect can be attributed to the higher inter element continuity of N-FEM. In literature a widely-used assumption is that 20 nodes per wavelength is sufficient in order to achieve good quality results [20] when applying conventional pure displacement finite elements to problems covering ultrasonic guided wave propagation. It can be seen that all proposed higher order finite element approaches need considerably less pseudo-nodes per wavelength reaching a minimum of only 3 when the proposed threshold is considered.
To evaluate the simulation results with respect to memory storage requirements the number of non-zero elements (nnz) of the system matrices $K$ and $M$ is determined for each polynomial degree when reaching the mentioned 1 % relative error threshold. The results are summarized in Fig. 4. All values are normalized with respect to the spectral element solution for the polynomial degree $p_{x1} = 2$. This serves as a measure of the memory storage requirements for each method. As can be inferred from Fig. 4 $p_{x1} = 3$ is optimal for all developed higher order finite elements. Due to their fast convergence isogeometric finite elements require the least degrees-of-freedom which results in a reduced demand of memory to store the system matrices. In that sense spectral finite elements and p-elements do not differ noticeably.

Fig. 3: Convergence curves (relative error vs. pseudo-nodes per wavelength) for the $A_0$-mode.

Fig. 4: Normalized memory storage requirements (normalized number of non-zero elements nnz) for the models reaching the relative error threshold of 1 %.
Discretization in transversal direction. After the optimal polynomial degree in longitudinal direction of the propagating wave has been determined a second convergence study is conducted to determine the optimal polynomial order in transversal direction. To ensure that the spatial discretization in longitudinal direction ($x_1$-direction) does not influence the quality of the results a polynomial degree of $p_{x1} = 6$ and a mesh density of $\chi_{A0} > 20$ is chosen (see Fig. 3e). In $x_2$-direction the discretization is comprised of one finite element with a varying polynomial degree $p_{x2} = 1, \ldots, 8$. The results of the convergence studies are displayed in Fig. 5.

![Fig. 5: Convergence curve for the $A_0$-mode for $p_{x1} = 6$ and $p_{x2} = 1, \ldots, 8$.](image)

As before, a relative error tolerance of 1 % is regarded as being sufficient. For all proposed methods a value of $p_{x2} = 3$ is adequate to meet the relative error criterion. Nonetheless, $p_{x2} = 4$ is chosen for all finite element approaches since $p_{x2} = 3$ only just meets the threshold.

Combining all results taken from Fig. 3a - 5 a guideline on how an appropriate discretization has to generated can be provided. A suitable polynomial degree template consists of $p_{x1} = 3$ and $p_{x2} = 4$ for all proposed higher order FE-schemes (H-FEM). With these polynomial orders the error threshold is met for both the $A_0$- and the $S_0$-mode. These templates are optimal in a sense that they provide for an efficient usage of the memory needed to simulate ultrasonic Lamb wave propagation problems. Fig. 6 provides results using this optimal polynomial degree template in combination with the number of pseudo-nodes per wavelength taken from the convergence curves of the discretization in longitudinal direction (see Fig. 3b). It displays the time history of the out-of-plane displacement field at the measuring points A and B in order to show that the amplitude of the signal is resolved as well as the group.

As can be seen in Fig. 6 the amplitude of the signal as well as the group speed is equally converged. A relative error of round about 1.7 % is reached for both quantities. If we take the area under the signal as measure to evaluate the accuracy of the numerical simulation a relative error of circa 2.5 % is reached. Studying these results one observes that the error threshold can be met deploying the discretization parameters, which can be extracted from the given convergence graphs (see Fig. 3a - 5).

![Fig. 6: Comparison of the results of the displacement component $u_1$ at points A and B computed using an analytical and an H-FEM (higher order FEM) approach. SEM: $p_{x1} = 3, p_{x2} = 4, \chi_{A0} = 7.09$; p-FEM: $p_{x1} = 3, p_{x2} = 4, \chi_{A0} = 7.52$; N-FEM: $p_{x1} = 3, p_{x1} = 4, \chi_{A0} = 4.09$.](image)
Comparison of higher order finite element approaches to conventional linear pure displacement elements. In the following the higher order finite elements are compared to conventional low order finite elements using a full integration or a selectively reduced integration to generate their element matrices. In order to avoid the occurrence of zero energy modes the latter are augmented with different types of hourglass control. In Fig. 7a the solutions obtained using the optimal polynomial degree template for the proposed higher order approaches are contrasted with standard low order finite elements. Three different methods to generate two-dimensional quadrilateral finite elements with 4 nodes are scrutinized. Type 1 is a fully integrated quadrilateral finite element without incompatible modes or other techniques to improve its convergence behaviour. Type 2 denotes a selectively reduced integrated quadrilateral finite element using the artificial stiffness method proposed in [21]. Type 3 is similar to the last finite element mentioned and differs only in the procedure to avoid zero energy modes. An enhanced hourglass control algorithm is chosen. This approach is based on the enhanced assumed strain and physical hourglass control methods proposed in [22]. For the conventional low order elements the aspect ratio varies only between 1 and 1.4. This measure takes into account that linear finite elements are prone to locking if the aspect ratio exceeds 1:2. Thus, the number of finite elements in transversal direction is variable in order to facilitate low aspect ratios. Since the $A_0$-mode resembles a flexural wave in essence the locking-effects are pronounced when dealing with fully integrated linear finite elements. For the reduced integrated elements the convergence rate of the $A_0$-mode is notably influenced by the reduced integration and the chosen hourglass control scheme. It is in general vastly improved and an appropriate choice of the hourglass control method renders linear finite elements competitive. Hence, the reduced integration of the shear energy term pays off and results in an improvement to $\gamma_{A0} = 8$ for the enhanced hourglass control algorithm (see Fig. 7a). The relative error is drastically reduced, although the accuracy of all higher order finite element approaches is still superior. Nevertheless, by using a selectively reduced integrated linear finite element the 1 % relative error bound is reached for a similar discretization as compared to the higher order schemes. Fig. 7b displays the same results for the first symmetric mode $S_0$. As can be seen, the hourglass control algorithm has virtually no influence on the accuracy and convergence rate of the symmetric mode.

Despite the improved convergence for the antisymmetric mode, certain disadvantages using conventional linear finite elements have to be addressed. Both the selectively reduced integration and the introduction of an hourglass control algorithm are intentional mistakes mathematically speaking. Since the reduced integration is focused on the shear energy term the whole system is softened and thus the convergence curves for linear finite element type 2 and 3 experience an offset. Due to the fact that two errors have been introduced to remedy the shortcomings of fully integrated finite elements the results of the modified system of equations do not converge to the exact solution of the mathematical problem. Additionally, there are numerous different approaches to introducing hourglass control methods, which are not all physically motivated. Thus, only by trial and error methods one can determine which of the methods is suitable for a given problem. Unlike this example, for more involved analyses there is almost never a reference solution to evaluate the results and thus to verify that the chosen hourglass control method is appropriate. Although, the convergence is vastly improved lower order finite elements are not recommended because of the mentioned drawbacks.
Conclusion

The suitability of three higher order finite element schemes to model the propagation of ultrasonic Lamb waves has been assessed in this article. In order to evaluate the quality of the solutions, the time of flights are computed using an analytical approach and are compared to the numerically obtained values. The results of the benchmark test have illustrated that higher order finite element methods are an efficient numerical tool and are essential when dealing with ultrasonic guided waves in thin-walled structures. The accuracy which can be achieved deploying these approaches is much improved compared to conventional finite element methods. Another important advantage is that no special measures have to be undertaken to avoid locking effects. Overall, the convergence properties of spectral finite elements and \( p \)-elements are very similar. Considering all results it has to be noted that \( p \)-elements promise the best accuracy, while isogeometric elements offer the highest convergence rates. An undisputable advantage of spectral elements is that because of the physical meaning of the degrees-of-freedom the pre- and post-processing is much more convenient. Additionally, mass-lumping techniques have already been developed and thus, explicit time integration schemes are a viable option for spectral elements. Regarding the results published in this paper, without considering the influence of the time integration scheme, isogeometric finite elements are recommended because of their high convergence rate.

In further studies the focus will be on the influence of mass lumping and time integration methods adjusted specifically for each higher order finite element approach, introduced in this article. Thus, an efficient numerical simulation tool with respect to computational time and memory requirements can be realized. In addition, the influence of the inter element continuity on the results is another interesting topic when dealing with isogeometric finite elements, which require further attention.

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