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Reply to RIVLIN's *Material symmetry revisited* or Much Ado About Nothing

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In a recent paper, Rivlin (2002) "revised" the concepts of material symmetry and strongly criticises the approach to symmetry and frame-indifference by Truesdell and Noll (1964). A simple comparison of these approaches, however, leads us to the conclusion that - apart from some minor and irrelevant differences - the two approaches have more in common than Rivlin concedes. To show this, we will briefly describe these two approaches and then show how they are related to each other.

1 The approach of Truesdell and Noll

Let $\kappa(X, t)$ be the motion of the body in a time interval $[t_0, t]$ and $\kappa_0(X)$ an (arbitrary) reference placement. The deformation gradient with respect to it is

$$\mathbf{F}(\mathbf{x}_0, t) := \mathrm{Grad}\boldsymbol{\chi}(\mathbf{x}_0, t) = \mathrm{Grad}\kappa(\kappa_0^{-1}(\mathbf{x}_0), t)$$

As all our considerations are local, we can drop the spatial argument $\mathbf{x}_0$ and write shortly

$$\mathbf{F}_t := \mathrm{Grad}(\kappa_t \bullet \kappa_0^{-1})$$

where the dot $\bullet$ stands for the composition of mappings. The general constitutive equation for the Cauchy stresses $\mathbf{T}$ of a simple material after Truesdell and Noll is assumed to be a functional of the history of the motion and the deformation gradient in that particular point

$$\mathbf{T} = \mathcal{F}|_{\tau=-\infty}^{t}\{\boldsymbol{\chi}_\tau, \mathbf{F}_\tau\}$$

The domain of this functional is not generally specified, and the question of which regularities are required of the processes is left open.

The use of such (half-infinite) histories is practical in the case of fading memory materials. However, for materials with permanent memory, this description is inadequate, as NOLL

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(1972) admits. For such materials, finite processes instead of histories are more appropriate, but require the introduction of a state space concept (s. NOLL, 1972, BERTRAM, 1982, 1989, 1993, 1999, SILHAVY, 1997).
Truesdell and Noll introduce the reference placement as invariant, in contrast to the motion, which is assumed to be objective under change of observer. The principle of material frame-indifference after Truesdell and Noll refers to the observer dependence of this functional and essentially consists of two parts.
Firstly, it is assumed that all observers use the same constitutive equation. If we consider two observers, one indicated by a superimposed asterisk and the other without, then this means that

$$\mathcal{F} = \mathcal{F}^* \tag{PFI}$$

in spite of the fact that both their values and their arguments are different. We will call this assumption principle of form-invariance (PFI).
Secondly, it is assumed that the action of a Euclidean transformation (TRUESDELL and NOLL 1964, Eq. 17.7)

$$\boldsymbol{\chi}_t^* = \mathbf{Q}_t\, \boldsymbol{\chi}_t + \mathbf{c}_t$$

on the CAUCHY stresses is a rotation, i. e. stresses are objective

$$\mathcal{F}|_{\tau=-\infty}^{t}\{\boldsymbol{\chi}_\tau^*, \mathbf{F}_\tau^*\} = \mathbf{Q}_t\, \mathcal{F}^*|_{\tau=-\infty}^{t}\{\boldsymbol{\chi}_\tau, \mathbf{F}_\tau\}\, \mathbf{Q}_t^T. \tag{PMO}$$

Here, $\mathbf{Q}_t$ is an arbitrary time-dependent orthogonal tensor. We will refer to this assumption as the principle of objectivity (PMO). This is also known as the principle of Euclidean frame indifference or the passive version of frame indifference in the literature. This principle seems to be universally accepted.
Truesdell and Noll also include a time-shift in this issue. But we will not repeat this here, as it seems to have no relevance in the controversy with Rivlin.
A third principle is the active version of the above, namely the principle of invariance under superimposed rigid body motions (PISM). It postulates that if one observer monitors two motions which are related by a Euclidean transformation, then the stresses are only rotated

$$\mathcal{F}|_{\tau=-\infty}^{t}\{\boldsymbol{\chi}_\tau^*, \mathbf{F}_\tau^*\} = \mathbf{Q}_t\, \mathcal{F}|_{\tau=-\infty}^{t}\{\boldsymbol{\chi}_\tau, \mathbf{F}_\tau\}\, \mathbf{Q}_t^T. \tag{PISM}$$

This postulate is the strongest of the three and means a clear restriction, as it excludes inertial sensitivity of the stresses. It has been objected to by many authors in the past, and in fact, counterexamples from rarified gases (e.g., MÜLLER, 1972) and turbulence show that it is not generally satisfied. On the other hand, it is always useful, as it gives rise to reduced forms.
Obviously, these three principles are not independent, but have the following logical structure

$$\begin{array}{lclcl} \text{PMO} & \wedge & \text{PISM} & \Rightarrow & \text{PFI} \\ \text{PMO} & \wedge & \text{PFI} & \Rightarrow & \text{PISM} \\ \text{PFI} & \wedge & \text{PISM} & \Rightarrow & \text{PMO} \end{array}$$

(s. SVENDSEN/BERTRAM, 1999, BERTRAM/SVENDSEN, 2001). In other words, two of them imply the third.
As a result of these principles, the stresses cannot depend on the motion χ_τ of the point, so that we can eliminate χ_τ from the list of arguments in the functional, for which we write now

$$\mathbf{T}_t = \mathcal{F}|_{\tau=-\infty}^{t}\{\mathbf{F}_\tau\}$$

by an abuse of notation. Besides this, the dependence on the deformation gradient cannot be arbitrary, as the rotations have to enter in a specific way. This leads to the usual reduced forms, which Rivlin also favours.
According to Truesdell and Noll, a symmetry transformation is a unimodular constant tensor $\mathbf{H}(\det\mathbf{H} = \pm 1)$, such that

$$\mathcal{F}|_{\tau=-\infty}^{t}\{\mathbf{F}_\tau\} = \mathcal{F}|_{\tau=-\infty}^{t}\{\mathbf{F}_\tau\,\mathbf{H}\} \qquad (\bar{\mathrm{S}})$$

holds for all histories of the deformation gradient. For solids, it is assumed that a so-called undistorted reference placement exists, with respect to which all symmetry transformations are orthogonal. If for a particular solid all orthogonal tensors are symmetry transformations, then the material is isotropic. In this case, we can apply representations of isotropic tensor functions in the usual way.
Utilizing these fundamental definitions and concepts, models have been formulated for many different kinds of material behaviour, both isotropic and anisotropic.

2 The approach of Rivlin

In contrast to Truesdell and Noll, Rivlin takes the initial placement κ_{t_0} as reference placement. This can be considered as a special case of Truesdell and Noll by identifying $\kappa_{t_0} \equiv \kappa_0$. However, when changing observer, Rivlin's reference placement is subject to Euclidean transformations, while Truesdell and Noll assume that it is invariant under change of observer. This fact becomes important when comparing invariance requirements (s. LIU, 2004), but should not lead to contradictions. Rivlin introduces his deformation gradient as

$$\mathbf{g}(\mathbf{x}_0, t) := \mathrm{Grad}\kappa(\kappa^{-1}(\mathbf{x}_0, t_0), t).$$

Below, we simply write

$$\mathbf{g}_t := \mathrm{Grad}(\kappa_t \bullet \kappa_{t_0}^{-1})$$

for this. In view of the chain rule, we have

$$\mathbf{F}_t = \mathbf{g}_t\,\mathbf{F}_{t_0}. \qquad (\mathrm{CR})$$

Rivlin's constitutive functional for the Cauchy stress tensor has the form

$$\mathbf{T}_t = \mathcal{G}|_{\tau=0}^{t}\{\mathbf{g}_\tau\}.$$

Rivlin thus uses finite processes instead of histories, but neither specifies mathematically the range of this functional, nor introduces a state space here.
The fundamental transformation utilized by Rivlin takes the form

$$\mathbf{Q}_t\ \mathcal{G}|_{\tau=0}^{t}\{\mathbf{g}_\tau\}\ \mathbf{Q}_t^T = \mathcal{G}|_{\tau=0}^{t}\{\mathbf{Q}_\tau\ \mathbf{g}_\tau\ \mathbf{Q}_{t_0}^T\}$$

for a generally time-dependent proper orthogonal tensor $\mathbf{Q}_t$. He introduces three postulates of this form

- the **symmetry transformation** by taking a constant $\mathbf{Q}_\tau \equiv \mathbf{S}$

$$\mathbf{S}\ \mathcal{G}|_{\tau=0}^{t}\{\mathbf{g}_\tau\}\ \mathbf{S}^T = \mathcal{G}|_{\tau=0}^{t}\{\mathbf{S}\ \mathbf{g}_\tau\ \mathbf{S}^T\} \tag{S}$$

- the **frame indifference** by taking $\mathbf{Q}_{t_0} \equiv \mathbf{I}$

$$\mathbf{Q}_t\ \mathcal{G}|_{\tau=0}^{t}\{\mathbf{g}_\tau\}\ \mathbf{Q}_t^T = \mathcal{G}|_{\tau=0}^{t}\{\mathbf{Q}_\tau\ \mathbf{g}_\tau\} \tag{FI}$$

- the **relative frame indifference** by taking $\mathbf{Q}_t \equiv \mathbf{I}$

$$\mathcal{G}|_{\tau=0}^{t}\{\mathbf{g}_\tau\} = \mathcal{G}|_{\tau=0}^{t}\{\mathbf{g}_\tau\ \mathbf{Q}_{t_0}^T\} \tag{rFI}$$

Note that no other symmetry transformations than proper orthogonal ones are permitted here. In both cases of frame indifference, it is understood that all observers use the same functional (PFI), so that the active and the passive version of frame indifference again coincide.

3 Relations between the two approaches

As stated above, the approach of Truesdell and Noll and that of Rivlin cannot be directly compared. Truesdell and Noll use (semi-infinite) histories, while Rivlin uses (finite) processes. And neither of these two approaches allows one to be embedded into the other. But this difference obviously has not played a role in the discussion so far. To make the two approaches more comparable, we consider the version of Truesdell and Noll's approach for finite processes obtained by truncating the histories prior to a certain time $\tau \equiv t_0$. Using the chain rule (CR), we then have the relation

$$\mathbf{T} = \mathcal{G}|_{\tau=0}^{t}\{\mathbf{F}_\tau\ \mathbf{F}_{t_0}^{-1}\} = \mathcal{F}|_{\tau=0}^{t}\{\mathbf{g}_\tau\ \mathbf{F}_{t_0}\}$$

In order to relate the invariance properties, we start with

$$\begin{aligned}\mathbf{Q}_t\,\mathcal{G}|_{\tau=0}^{t}\{\mathbf{g}_\tau\}\,\mathbf{Q}_t^T &= \mathcal{G}|_{\tau=0}^{t}\{\mathbf{Q}_\tau\,\mathbf{g}_\tau\} \\ &= \mathbf{Q}_t\,\mathcal{G}|_{\tau=0}^{t}\{\mathbf{F}_\tau\,\mathbf{F}_{t_0}^{-1}\}\,\mathbf{Q}_t^T = \mathcal{G}|_{\tau=0}^{t}\{\mathbf{Q}_\tau\,\mathbf{F}_\tau\,\mathbf{F}_{t_0}^{-1}\} \\ &= \mathbf{Q}_t\,\mathcal{F}|_{\tau=0}^{t}\{\mathbf{g}_\tau\,\mathbf{F}_{t_0}\}\,\mathbf{Q}_t^T = \mathcal{F}|_{\tau=0}^{t}\{\mathbf{Q}_\tau\,\mathbf{g}_\tau\,\mathbf{F}_{t_0}\} \\ &= \mathbf{Q}_t\,\mathcal{F}|_{\tau=0}^{t}\{\mathbf{F}_\tau\}\,\mathbf{Q}_t^T = \mathcal{F}|_{\tau=0}^{t}\{\mathbf{Q}_\tau\,\mathbf{F}_\tau\}\end{aligned} \tag{FI}$$

which is essentially PISM.
In order to relate the symmetry transformations, we start with

$$\begin{aligned}\mathbf{S}\,\mathcal{G}|_{\tau=0}^{t}\{\mathbf{g}_\tau\}\,\mathbf{S}^T &= \mathcal{G}|_{\tau=0}^{t}\{\mathbf{S}\,\mathbf{g}_\tau\,\mathbf{S}^T\} \\ &= \mathbf{S}\,\mathcal{F}|_{\tau=0}^{t}\{\mathbf{g}_\tau\,\mathbf{F}_{t_0}\}\,\mathbf{S}^T = \mathcal{F}|_{\tau=0}^{t}\{\mathbf{S}\,\mathbf{g}_\tau\,\mathbf{S}^T\,\mathbf{F}_{t_0}\} \\ &= \mathbf{S}\,\mathcal{F}|_{\tau=0}^{t}\{\mathbf{F}_\tau\}\,\mathbf{S}^T\end{aligned} \tag{S}$$

and by use of PISM, one obtains

$$\begin{aligned}\mathbf{S}\,\mathcal{F}|_{\tau=0}^{t}\{\mathbf{F}_\tau\}\,\mathbf{S}^T &= \mathcal{F}|_{\tau=0}^{t}\{\mathbf{S}\,\mathbf{F}_\tau\} \\ &= \mathcal{F}|_{\tau=0}^{t}\{\bar{\mathbf{F}}_\tau\} = \mathcal{F}|_{\tau=0}^{t}\{\bar{\mathbf{F}}_\tau\,\mathbf{H}\}\end{aligned} \tag{$\bar{\text{S}}$}$$

via the identifications

$$\begin{aligned}\bar{\mathbf{F}}_\tau &\equiv \mathbf{S}\,\mathbf{F}_\tau \\ \mathbf{H} &\equiv \mathbf{F}_{t_0}^{-1}\,\mathbf{S}^T\,\mathbf{F}_{t_0}\end{aligned}$$

the latter of which is known as NOLL's rule (of the transformation of the symmetry group under change of reference placement). Note that Truesdell and Noll's symmetry transformation need not be orthogonal (just unimodular), while Rivlin makes this restriction, by which he limits his applications to solids with respect to undistorted initial placements.
But even in this case, the question remains open as to whether symmetry transformations have to be proper orthogonal or just orthogonal. For some theories (e.g., involving axial vectors like the magnetic field flux), this is an important physical question. In the present context of simple mechanical materials, however, it turns out that only tensors of even order are related, so that a change in sign would generally play no role.
So, apart from these details, we have the inclusions

$$\mathrm{FI} \;\Leftrightarrow\; \mathrm{PISM}$$

and

$$\mathrm{SI} \;\wedge\; \mathrm{PISM} \;\Leftrightarrow\; \overline{\mathrm{S}}.$$

Thus, Rivlin's and Truesdell and Noll's forms of frame-indifference are essentially equivalent. And if we assume their validity, also both symmetry definitions coincide for solids.

It is not clear to the present authors what Rivlin had in mind when he postulated rFI. As FI together with rFI leads to isotropy (Rivlin p.115), this additional postulate remains questionable. Frame indifference and material symmetry are conceptually distinct, such that one can never be deduced from the other when properly formulated.

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