The torsional buckling of a cruciform column under compressive load with a vertex plasticity model

M. Schurig a,*, A. Bertram b

Abstract

The torsional buckling of a plastically deforming cruciform column under compressive load is investigated. The problem is solved analytically based on the von Kármán shallow shell theory and the virtual work principle. Solutions found in the literature are extended for path-dependent incremental behaviour as typically found in the presence of the vertex effect that is present in metallic polycrystals.

At the critical load for buckling the direction of straining changes by an additional shear component. It is shown that the incremental elastic–plastic moduli are spatially nonuniform for such situations, contrary to the classical \( J_2 \) flow and deformation theories. The critical shear modulus that governs the buckling equation is obtained as a weighted average of the incremental elastic–plastic moduli over the cross-section of the cruciform.

Using a plasticity model proposed by the authors, that includes the vertex effect, the buckling-critical load is computed for an aluminium column both with the analytical model and a FEM-based eigenvalue buckling analysis. The stable post-buckling path is determined by the energy criterion of path-stability.

1. Introduction

Thin walled open structures tend to buckle in the torsion mode under applied compressive loads. If the applied load exceeds the yield load, the twisting structure remains in the plastic state in the whole cross-section. This is in contrast to slender columns that buckle in a bending mode. Here, part of the cross-section unloads.

Moreover, bending is a completely one-dimensional case without any influence of the vertex effect. In the torsion mode, however, the flanges of the column show additional twisting and thus change from compression to a combination of compression and shear.

The plastic cruciform is an example for the failure of the yield surface at the current stress point. For total loading histories, integration of distinct slip theories is possible leading to deformation theories of plasticity (Sanders, 1954; Hutchinson, 1974; Stören and Rice, 1975).

When directly comparing \( J_2 \) flow theory and deformation theories for buckling of different shell structures has been compared by Bushnell (1982).
Structures buckling in the plastic regime exhibit a strong imperfection-sensitivity (Onat and Drucker, 1953). Using a nonlinear von Kármán shell theory with an inelastic incremental law, Hutchinson (1974), Hutchinson and Budiansky (1976) showed that the reason for the high predicted buckling load is the shear stiffness in a pre-compressed cruciform column. The $J_2$ flow theory predicts elastic behavior after such a change of the strain path while the deformation theory neglects the possibility of unloading. However, by introducing an imperfection (pretwist) of the column, useful results could be obtained.

Alternative material models provide a solution to the buckling paradoxon, as both the $J_2$ deformation and flow theory neglect the interaction of a multitude of distinct slip mechanisms.

A $J_2$ corner theory has been proposed by Christoffersen and Hutchinson (1979). This theory is based on physically justified assumptions and has nearly proportional strain occurring everywhere similar to the deformation theory (Hutchinson and Tvergaard, 1980; Cimetinre et al., 2004). It is based on a rate-type material model and a transition function that acts as a factor on the plastic part of the incremental moduli tensor in the rate-potential for process paths deviating from a proportional process continuation.

In this article, a material model proposed by Schurig and Bertram (submitted for publication) for vertex plasticity has been applied to the cruciform column. It is based on finite elastic–plastic behavior and models the interaction of several plastic mechanisms on the microscale in the flow rule without application of a complex homogenization scheme.

Accordingly, the vertex effect is not described by its implications on the incremental moduli tensor but in the direction and magnitude of the plastic flow that has been made dependent on the process continuation. This requires that the derived plastic part of the incremental moduli tensor has a rank that exceeds 1 as found in crystal plasticity but not in the $J_2$ theory with the normality rule. The constitutive equations are able to predict the stress–strain behaviour of a numerical polycrystal model immediately after a strain path change from which the model functions have been identified. The transition from total loading to total unloading is also reproduced in the dissipated fraction of stress power. Moreover, in comparison to the $J_2$ flow theory the prediction of strongly non-proportional tension–torsion experiments can be improved without much increased computational complexity. $J_2$ deformation theory is not usable for such problems. If the deformation theory is identified with total loading behaviour, a lower bound for the buckling critical loads can be obtained. However, for the secondary path, the according flow theory has to be used that relies upon artificial imperfections for triggering (Hutchinson and Budiansky, 1976). The amplitude of such imperfections is not a given geometrical parameter but must be treated as an additional parameter that needs to be identified appropriately (Bardi et al., 2006; Corona et al., 2006).

A reduced incremental modulus has also been predicted by an generalized visco-plastic material model (Rubin and Bodner, 1995).

### 1.1. Article overview

In Section 2, the governing structural equations for the buckling analysis of a cruciform column are developed following Hutchinson and Budiansky (1976), using von Kármán’s shell theory and the virtual work principle. The form presented here allows for insertion of the rate form of the proposed vertex material model.

In Section 3, in contrast to previous treatments, the equation for the reduced modulus is obtained for a plastic shear modulus that varies over the shell thickness according to the vertex effect. The strain rate and direction parameters necessary for the vertex material model are obtained from the assumed buckling kinematics. The result is a suitable weighted through-thickness average of the plastic shear modulus. Results from the literature are obtained as a special case.

In Section 4 the secondary path after buckling is obtained by application of the energy criterion of path stability (Petryk, 1991) for the proposed vertex material model.

For validation, the classical experimental results by Gerard and Becker (1957) are compared with the closed-form solution presented here and a FEM-based buckling solution using the same vertex material model, Sections 5.1, and 5.2.

### 1.2. Notation

Second order tensors and vectors are denoted by upper- and lowercase boldface Latin characters $A$, $V$, respectively. The dyadic product is written as $A \otimes V$. Single contraction is denoted by juxtaposition of the according symbols $A B$, $AV$. The scalar product between tensors is written with a single dot $A \cdot B = \text{tr}(AB^T)$. Fourth order tensors are denoted like $A$ and the application to second order tensors by $A[B]$. $A \circ B[A]$ is used shorthand for $\mathcal{A}(B)$. $\mathcal{A}$ is the traceless part of $A$ (deviator). The Rayleigh product $P \star A$ is a shorthand notation for the tensor having the components $p_{abpcdpefpghabdfh}$.

### 2. Buckling of an elastic–plastic plate

We model the behaviour of the cruciform column shown in Fig. 1(a) by the nonlinear von Kármán plate theory as proposed by e.g., Hutchinson and Budiansky (1976), Nguyen (2000). The presentation given in the sequel is valid for shallow shells as well.

Let $x_i$ and $x_2$ be the in-plane coordinates that parameterize the shell’s mid-surface, while $x_3 \equiv z$ is the thickness coordinate. Accordingly, $x_i = (x_1, x_2)$ is the planar position vector component. For differentiations, we use the planar Nabla operator $\nabla_2$.

The displacement vector is decomposed in the same way, $u_i = (u_1, u_2, 0) \equiv w$. If the reference coordinates of the shell are given by $Z(x_1, x_2)$, the planar von Kármán strain acting in the mid-surface is

$$E_i^2(x_2) = \varepsilon_2 + \frac{1}{2} \left( \nabla_2 Z \otimes \nabla_2 w + \nabla_2 w \otimes \nabla_2 Z + \nabla_2 w \otimes \nabla_2 w \right).$$

(1)

Plate kinematics are introduced using the tensor of curvature

$$K(x_2) = \nabla_2 w \otimes \nabla_2$$

(2)

and the Bernoulli–Kirchhoff hypothesis resulting in the strain

$$E_i(x_2, z) = E_i^2(x_2) - zK(x_2).$$

(3)

For the torsional displacement of a single flange in the $x_1 - x_2$-plane the ansatz

$$w(x_2) = x_2 \phi(x_1),$$

(4)

(Fig. 1(b)) with an unknown function $\phi$ of the longitudinal coordinate $x_1$ is appropriate, and a similar one for a torsional imperfection mode,

$$Z(x_2) = x_2 \Phi(x_1).$$

(5)

For the in-plane displacement, Hutchinson and Budiansky (1976) made the following proposal:

$$u_1 = c_0 x_1,$n

$$u_2 = - \left( \Phi \phi^2 + \phi^2 - c_1 \right) x_2.$$n

(6)

The in-plane shear and the 22 components of the linear and nonlinear part cancel each other, and the midsurface strain tensor is
The according tensor of curvature $K = \phi''x_2 \phi'\phi - \phi'z\phi' - \phi''z\phi'$, gives the planar Green's strain field in the von Kármán approximation $E = \epsilon_0 + (\phi\phi' - \phi''x_2\phi' - \phi''z\phi' - \phi'z\phi' \epsilon_1)$. By introducing the torsion rate $u = \dot{\phi}$, the according strain rate field is $E = \epsilon_0 + (\phi\phi' - \phi''x_2\phi' - \phi''z\phi' - \phi'z\phi' \epsilon_1)$. Application of a plain strain incremental material law yields the second Piola–Kirchhoff stress rate, for the rates $\epsilon_0$ and $\phi$, following Hutchinson and Budiansky (1976) we use the virtual work principle in the presence of the normal force $N_{11} = \frac{f}{2} S_{11} dz$ and the twisting moment $M_{12} = \frac{f}{2} S_{12} dz$, where the $S_{ij}$ are the according components of the second Piola–Kirchhoff stress, time differentiation of this material equation yields a rate equation for the rates $\epsilon_0$ and $\phi$, two independent results can be obtained.
\[
\begin{align*}
\int_0^1 \int_{-1}^1 S_{11} dz dx_2 + \bar{p} &= 0, \\
\int_0^1 \int_{-1}^1 \left[ -2S_{12} z + S_{11} x_2^2 (\phi') + S_{11} x_2^2 (\bar{\Phi} + \Phi') \right] dz dx_1 &= 0.
\end{align*}
\]

(15)

Starting from a undisturbed placement on the primary path as given by \( \bar{\Phi} \equiv 0 \), and using \( S_{12} = K_{33} E_{12} = -K_{33} Z \phi' \), the bifurcation equation

\[
\int_0^1 \int_{-1}^1 \left[ 2K_{33} z^2 + S_{11} x_2^2 \right] \phi' \delta \Phi' dz dx_1 = 0,
\]

(16)

has a solution which is independent of the actual shape of the eigenmode \( \phi \).

3. The reduced modulus

The choice of the modulus \( K_{33} \) has been of major interest to research, due to the finding that the elastic shear modulus as predicted by the \( J_2 \) flow theory overestimates the experimental results.

(10) shows that for compression (\( \phi = 0 \)) followed by the secondary post-buckling path (\( \phi \neq 0 \)), the direction of the strain rate tensor changes. A material model with vertex effect for such processes has been proposed by Schurig and Bertram (submitted for publication).

Using the formulation outlined by Bertram (1999), Bertram (2008), a linear elastic law for the second Piola–Kirchhoff stress tensor \( S \) used for an evolving elastic–plastic material with isomorphic elastic ranges has the structure

\[
S(C) = PC \left[ \frac{1}{2} [P^* CP - 1] \right] P^T.
\]

(17)

Here \( \bar{C} \) is the fourth-order tensor of elasticity, \( C = F \bar{C} F \) is the right Cauchy-Green-Tensor, while \( P \) is the plastic transformation. \( P \) is constant in purely elastic processes. In plastic processes, however, it evolves according to a flow rule.

(17) can be reformulated to derive a multiplicativ decomposition by the definitions

\[
\begin{align*}
F_p &= P^{-1}, \\
F_e &= FP,
\end{align*}
\]

resulting in \( F = F_p F_e \) and \( C_e = F_e^{-1} F_p \) and \( S_e = P^{-1} S P^{-1} \).

Starting with the Huber–von-Mises yield criterion expressed by a Mandel-type stress tensor,

\[
\bar{\Phi}(C, S_e, g) = \frac{3}{2} \left[ \left( C S_e \right)' \cdot (C S_e)' - g \right].
\]

(19)

where \((\cdot)'\) denotes the deviatoric part of a tensors, a plastic potential that accounts for the vertex effect can be conveniently formulated using the base tensors \( B_e \) and \( B_a \) (see Fig. 2),

\[
\bar{\Psi}(C, S_e, g, B_a, \bar{M}) = F(x) \bar{\Phi} + (1 - F(x)) B_a \cdot (C S_e)' = 0.
\]

(20)

In Schurig (2006), Schurig and Bertram (submitted for publication) the activity of the slip systems existing in the crystallites has been connected with the normalized tensor

\[
\bar{M} = \frac{\partial \bar{\Phi}(C S_e, g, B_a, \bar{M})}{\partial \bar{\Phi}} = \frac{\partial \bar{\Psi}(C, S_e, g, B_a, \bar{M})}{\partial \bar{\Psi}}.
\]

(21)

The direction with total loading (the natural direction (Takahashi et al., 1990)) is denoted by \( \bar{M} = B_a \). Here it is identified with the direction of proportional straining.
For small elastic strains, the resulting incremental elastic–plastic law is
\[ \mathbf{S} = \mathbf{P} \star \mathbf{\kappa} [\mathbf{E}], \]  
(25)
containing the incremental elastic–plastic moduli tensor,
\[ \mathbf{\kappa} = 2G\left(\mathbf{I} + \frac{\mathbf{V}}{1-2\nu} \otimes \mathbf{1} \right) = F_1(x) \]
\[ \times \frac{4G^2(F(x)B_0 + (1-F(x))B_a) \otimes (F(x)B_0 + (1-F(x))B_a)}{2G(F(x)^2 + (1-F(x))^2) + F^2(x)\mathbf{\eta}^T \mathbf{\eta}}. \]  
(26)

Here, the difference to, e.g. Christoffersen and Hutchinson (1979) can be found in the plastic part (26.2) that changes its magnitude and tensorial direction according to the direction parameter \( \alpha \).

The plane stress moduli needed for the buckling analysis can be obtained by
\[ \mathbf{\kappa}_2 = -\mathbf{\kappa} [\mathbf{e}_2 \otimes \mathbf{e}_2] \otimes \mathbf{\kappa} [\mathbf{e}_2 \otimes \mathbf{e}_2], \]  
(27)
where \( \mathbf{e}_2 \) is the unit vector in through-thickness direction of a plate.

For the proposed vertex model, the incremental moduli rely on the hardening modulus \( H = \frac{0.6}{\mathbf{\mu}} \) and the loading parameter. Figs. 3 and 4 show the incremental moduli, normalized by their respective elastic values.

The longitudinal modulus increases steadily with deviation from a proportional process (\( \alpha = 1 \)) and finally enters the elastic value for the reversed process (\( \alpha = -1 \)). The shear modulus shows a different behaviour. In addition to the reversed process, for \( \alpha = 1 \) the elastic modulus is obtained. This seems to be a parallelism to the \( J_2 \) flow theory. However, one should bear in mind that after a tensile process, shear cannot be obtained in a process of \( \alpha = 1 \). For a total shift to a shear processes, \( \alpha = 0 \) and, accordingly, the incremental modulus of the vertex model is much lower in comparison to \( J_2 \) flow theory.

In the proportional fundamental (primary) path before buckling occurs, the natural direction is given by
\[ \mathbf{\dot{B}}_1 = \frac{1}{\sqrt{\frac{1}{2}(\dot{\bar{\varepsilon}}_0^2 + \dot{\bar{\varepsilon}}_0 + \dot{\bar{\varepsilon}}_1^2)}} \left( \begin{array}{ccc} -\frac{1}{2} \dot{\bar{\varepsilon}}_0 - \frac{1}{2} \dot{\bar{\varepsilon}}_1 & 0 & 0 \\ 0 & +\frac{1}{2} \dot{\bar{\varepsilon}}_0 + \frac{1}{2} \dot{\bar{\varepsilon}}_1 & 0 \\ 0 & 0 & \dot{\bar{\sigma}}_0 - \dot{\bar{\varepsilon}}_1 \end{array} \right). \]  
(28)

The additional twisting mode of the secondary path from (10) gives the strain rate deviator
\[ \mathbf{E}' = \left( \begin{array}{ccc} -\frac{1}{2} \dot{\bar{\varepsilon}}_0 - \frac{1}{2} \dot{\bar{\varepsilon}}_1 & 0 & 0 \\ 0 & +\frac{1}{2} \dot{\bar{\varepsilon}}_0 + \frac{1}{2} \dot{\bar{\varepsilon}}_1 & 0 \\ 0 & 0 & \dot{\bar{\sigma}}_0 - \dot{\bar{\varepsilon}}_1 \end{array} \right) \]  
(29)
and accordingly
\[ \mathbf{M} = \frac{1}{\sqrt{\frac{1}{2}(\dot{\bar{\varepsilon}}_0^2 + \dot{\bar{\varepsilon}}_0 + \dot{\bar{\varepsilon}}_1^2 + 2z^2 \dot{\bar{\varphi}}^2) + 2z^2 \dot{\bar{\varphi}}^2}} \times \left[ \begin{array}{ccc} -\frac{1}{2} \dot{\bar{\varepsilon}}_0 - \frac{1}{2} \dot{\bar{\varepsilon}}_1 & 0 & 0 \\ 0 & +\frac{1}{2} \dot{\bar{\varepsilon}}_0 + \frac{1}{2} \dot{\bar{\varepsilon}}_1 & 0 \\ 0 & 0 & \dot{\bar{\sigma}}_0 - \dot{\bar{\varepsilon}}_1 \end{array} \right] \]  
(30)
Thus, the loading parameter \( \alpha \) varies in the cross-section. With \( \mu = \frac{\dot{\bar{\tau}}_0}{\dot{\bar{\varepsilon}}_0} \).
For a uniaxial deformation process, \( \mu = \epsilon_1 = 0 \), and
\[
\alpha = \frac{1 + \mu + \mu^2 - 2\mu J_{0}\omega'}{\sqrt{1 + \mu + \mu^2 + 2\mu J_{0}\omega'^2 + 3\mu^2 (\omega')^2}}. \tag{31}
\]

Hutchinson and Budiansky (1976) proposed a linear \( x_1 \)-dependence of the bifurcation eigenmode
\[
\phi' = \Theta \Rightarrow \phi'' \equiv 0, \quad \phi = \Theta, \tag{33}
\]
(see Fig. 1(b)), which further reduces the result to
\[
\alpha = \frac{1}{\sqrt{1 + 3(\omega')^2}}. \tag{34}
\]

This arbitrariness is justified by the shape-independence of the bifurcation Eq. (16).

The only remaining variability is in \( z \)-direction. It has been plotted in Fig. 5 for different values of \( \frac{z}{T} = 2^q \), \( q = 0 \ldots 8 \). Obviously, on the midsurface \( z = 0 \) we obtain \( \alpha = 1 \). There is a range in the vicinity of the midsurface where the \( J_2 \) flow theory (which is characterized exactly by \( \alpha = 1 \)) is a good approximation. Its thickness shrinks with growing dominance of the buckling mode. For the limit case, we obtain
\[
\lim_{\frac{z}{T} \to \infty} \alpha = \begin{cases} 1 & \text{at } z = 0 \text{ only}, \\ 0 & \text{almost everywhere}. \end{cases} \tag{35}
\]

We integrate the essential part of (16)_2,
\[
\int_0^1 \int_{\frac{z}{T}}^{\frac{1}{2}} [2K_{33}z^2 + S_{11}x_1^2]dzdx_2 = \frac{BT^2}{6}K_{33} + S_{11}TB^3 \frac{3}{2} = 0, \tag{36}
\]
where the reduced shear modulus \( K_{33} \) has been obtained by application of the mean value. It can be explicitly obtained by substitution,
\[
z = (3u)^\frac{1}{3}, \quad \alpha = \frac{1}{\sqrt{1 + 3(\omega')^2 (\omega')^2}}, \tag{37}
\]
\[
K_{33} = \frac{24}{T^2} \frac{1}{3} \int_0^1 K_{33}(\alpha)du.
\]

The result for the critical buckling load is
\[
S_{11} = \frac{1}{2} P_{33} \left( \frac{T}{B} \right)^2. \tag{38}
\]

For \( z \)-independent modulus, \( K_{33} = K_{33} \). The latter is the classical result. It has been used with different choices for the modulus \( K_{33} \).

Gerard and Becker (1957) preferred the incremental modulus from \( J_2 \) deformation theory due to its better fitting of experimental results than the elastic shear modulus. Hutchinson (1974) explained this fact by the possibility to describe total loading behaviour by the deformation theory. Hutchinson and Budiansky (1976) used the \( J_2 \) flow theory in conjunction with a preexisting imperfection that cured the problem raised by the excessive lateral stiffness by application of the plastic tangential modulus. Based on such experience, Bažant and Cedolin (1991, Section 8.1) propose to use the tangential modulus on the secondary path for structural buckling, i.e. an a-priori assumption.

The exact function for \( K_{33} \) has been integrated numerically for different values of \( \frac{z}{T} \) and \( T \) at perfect plastic (\( g = \text{const} \)) behaviour and can be found in Fig. 6.

With increasing influence of the buckling mode, the reduced shear modulus decreases. Fig. 5 shows, that the lowest possible value can be obtained in the limit for an infinite ratio \( \frac{z}{T} \) leading to \( \alpha = 0 \) almost everywhere. An increasing thickness of the shell leads to smaller reduced moduli. Fig. 6 does not agree with neither the \( J_2 \) flow nor the deformation theory that both predict a reduced shear modulus that does not depend on the twist. However, for the limit case, the lowest possible value of \( K_{33} \) is attained. This may be the explanation for the good agreement with the deformation theory and its inherent assumption that the transition from compression to shear remains total loading with the associated low value of all incremental elastic–plastic moduli.

4. Determination of the secondary path by the rate functional

In analogy to the three dimensional theory, the virtual work principle is transferred into a rate problem which is submitted to Petryk's energy criterion of path stability (Petryk, 1991). It results in a selection criterion for the secondary path and allows for a final exploitation of the buckling equation (38).

For rate-independent materials with a rate potential, insertion of the incremental material law (27) starting from the primary path without torsional displacement results in the rate functional
\[
H(\dot{\epsilon}_0, \varphi) = \int_0^1 \left( \int_0^\frac{1}{3} \int_{\frac{z}{T}}^{\frac{1}{2}} \left( 2K_{33}z^2 + S_{11}x_1^2 \right) \right) \frac{(\varphi')^2}{2} + K_{11} \frac{\dot{\varphi}^2}{2} \left( \frac{d}{dx_1} + \frac{\dot{\rho}_0}{T} \right) \right) dx_1. \tag{39}
\]

The reduced shear modulus \( K_{33} \) as a function of \( \frac{z}{T} \) at different thickness values \( T = 0.1 \text{ mm (upper)} \) and \( T = 0.2 \text{ mm (lower line)} \).
\[
\frac{1}{2} \Delta \mathcal{E} = \frac{d^2}{dt^2} \int_0^t \left( \int_0^t \left( 2M_{12}K_{12} + N_{11}(E_{21}) \right) dx_2 + P\dot{\epsilon}_0 \right) dx_1 dt \\
= H(\epsilon_0, \varphi).
\]  
(40)

Accordingly, processes that do not minimize the rate functional but give only a saddle point are not stable (Petryk, 1991; Petryk, 2000; Fedelich and Ehrlacher, 1997).

The variation with respect to the two variables \( \dot{\epsilon}_0, \varphi(x_1) \) yields (14). By the previously used ansatz for the eigenmode, \( \varphi' = \dot{\theta} \), the integration results after division by \( L \)

\[
H(\dot{\epsilon}_0, \theta) = \frac{BT}{2} \left[ S_{11} + \frac{1}{2} \left( \frac{\dot{T}}{B} \right)^2 \left( K_{33} \right) \right] \dot{\theta}^2 + BTK_{11}\dot{\epsilon}_0 + P\dot{\epsilon}_0.
\]  
(41)

Here, a reduced longitudinal modulus

\[
K_{11} = \frac{1}{T} \int_{-\Delta}^{\Delta} K_{11} dz.
\]  
(42)

has been introduced similar to (37).

By minimization, we calculate the partial derivatives

\[
\begin{align*}
H_{\dot{\epsilon}_0} &= BTK_{11}\dot{\epsilon}_0 + P = 0, \\
H_{\dot{\theta}} &= \frac{BT}{2} \left[ S_{11} + \frac{1}{2} \left( \frac{\dot{T}}{B} \right)^2 \left( K_{33} \right) \right] \dot{\theta} = 0.
\end{align*}
\]  
(43)

leading to a minimum at \( \dot{\theta} = 0, BTK_{11}\dot{\epsilon}_0 = -\dot{P} \). This is exactly the primary path.

The minimum exists, if the second variation is positive, i.e. the bracketed term is positive. At the critical load, the minimum and the related equilibrium state get indefinite. For a further increasing compressive load an instable process results.

We represent the rate functional in a normalized way,

\[
H(x, y) = \frac{BT}{R^2} \left[ \frac{S_{11}}{3} (1 - Q) \frac{x^2}{2} + K_{11} \left( \frac{y^2}{2} + y \right) \right].
\]  
(44)

\[
S_{11} = \frac{1}{2} (\ddot{\theta})^2 K_{33}, \quad Q = -\frac{s_{11}}{s_{66}}, \quad R = \frac{E}{\rho}, \quad x = B\dot{\theta}, \quad y = R\dot{\epsilon}_0.
\]

The parameter \( Q \in \{0.9, 1, 1.1\} \) selects examples of subcritical, critical and postcritical loads. At the critical point, the quadratic form changes from the elliptic to the hyperbolic range, supporting the above finding of lost stability. Fig. 7 shows contour lines of the normalized rate functional for the mentioned cases, assuming a constant value of \( K_{11} \) and of \( R_{\theta} \), which corresponds to replacing \( H \) by its second variation.

In the critical case, an indifferent situation can be found, assigning a minimum to every path with \( y = -1 \) \( \Rightarrow BTK_{11}\dot{\epsilon}_0 = -\dot{P} \).

When considering the dependence of \( K_{11} \) on \( x \) and thus on \( \frac{\dot{\theta}}{\dot{\epsilon}_0} = \frac{1}{\dot{\epsilon}_0} \), the picture differs somewhat. The rate potential at the critical load can be written as

\[
H' = \frac{BTK_{11}}{R^2} \left( \frac{\dot{\theta}^2}{2} + y \right).
\]  
(45)

Accordingly, the minimum of \( f(x) \) is reached by an independent adjustment of \( \frac{1}{\dot{\epsilon}_0} \) at a given rate of the applied force, \( \dot{P} \).

As shown in Fig. 3, \( K_{11} \) grows with decreasing \( x \). Thus, the limit state

\[
\begin{align*}
(a) \quad Q &= -0.9 \text{ Subcritical} \\
(b) \quad Q &= -1 \text{ Critical} \\
(c) \quad Q &= -1.1 \text{ Postcritical}
\end{align*}
\]
reaches the minimum almost everywhere. As a consequence, \( K_{11} = K_{13}, K_{33} = K_{13}. \)

For the postcritical case, the minimum transforms into a saddle point, which is a typical result for buckling problems. Such results have also been reported for other structural stability problems, namely the discrete Shanley’s column (Petryk, 1991).

5. Results and discussion

5.1. Comparison with buckling experiments

To assess the capability of the proposed vertex model, the minimizing incremental shear modulus has been inserted into (38),

\[
S_{11} = \frac{1}{2} K_{11}(\alpha = 0) \left( \frac{T}{B} \right)^2.
\]

The experimental results reported by Gerard and Becker (1957) for 2024-T4 Aluminium by plotting the buckling reduction factor

\[
\eta(S_{11}) = \frac{S_{11}}{S_{el}} = \frac{K_{11}}{K_{11}^{el}},
\]

show the reduction of the critical stress compared to the elastic theory. It is easily obtained if the reduced shear modulus \( K_{11} \) is given. As noted before, the \( J_2 \) flow theory is unable to reproduce this difference.

To evaluate (48), an analytical expression has been adopted for the hardening behaviour, as proposed by Papadopoulos and Lu (1998).

\[
g(\alpha) = \sigma_y^0 + q(\alpha),
q(\alpha) = H_0 \alpha + (\sigma_y^0 - \sigma_y^0)(1 - \exp(-\alpha)).
\]

The graph has been shown in Fig. 8 and the parameters can be found in Table 2. For different values of \( \alpha \), (48) has been evaluated. These values are typically in the transitory nonlinear hardening range.

To compare the results with the experimental observations, the experimental data from Gerard and Becker (1957) have been digitized and the units converted to the SI system. In addition, clusters of repeated tests with similar results have been identified. For each cluster, the mean value and the standard deviation have been computed. Fig. 9 reports this data, together with the respective result of the vertex model.

The latter exhibits a hardening-dependent reduction of the critical buckling load compared to the elastic response. The theoretical prediction in the plastic regime (for which the line has been drawn) is within the double standard deviation around the centers of gravity of the experimental clusters. As further experimental data is lacking, an extrapolation to a larger range of strain has not been done.

From the structure of the governing Eqs. (48), (49), it is clear that in the linear hardening range the reduced modulus is constant, leading also to a constant value for the buckling reduction factor.

5.2. Numerical approach

A numerical treatment of the cruciform column based on the torsion dominant incremental moduli \( \alpha = 0 \) and eigenvalue buckling in an iterative approach confirms the above results.

5.3. The structural model

The cruciform column has been modelled with the finite element code ABAQUS. Bi-linear four-node shell elements with reduced integration (S4R) elements have been used. As for the boundary conditions, each flange has been modeled as simply supported at the bottom end. The upper end edges have been constrained to follow the rigid body motion of the center line end point which is additionally constrained excluding rotations except in torsion mode. Thus bending-like buckling has been prevented. At the upper edges a uniform compressive line load has been applied. A typical mesh is shown in Fig. 10.

5.4. Eigenvalue buckling

For the numerical approach, an equivalent to the bifurcation Eq. (36) has to be found.

A solution for a material of the rate type is supplied by the eigenvalue buckling analysis. For a prescribed unit load case, the load factor can be obtained from the general eigenvalue problem of the incremental system stiffness matrix.
are independent of the particular choice of sorial base f.

The rotational symmetric incremental stiffness matrix, the latter being proportional to the applied external load factor a. Being a line load, it determines the uniaxial stress $\sigma_x = \frac{F}{A}$.

$K_0$ and $\Delta K_0$ are the dynamic and geometric parts of the system stiffness matrix, the latter being proportional to the applied external load factor. Besides the definition of geometry and load case, the user has to supply the appropriate incremental stiffness tensor.

According to the vertex model, all directions orthogonal to $B_n$ are equivalent, as the theory automatically adjusts to each of them. Such complicated computations are impossible with the eigenvalue buckling analysis where the secondary direction and thus $B_n$ are not known in advance. From (26),

$$K_p \propto (F(x)B_a + (1 - F(x))B_b) \otimes (F(x)B_a + (1 - F(x))B_b)$$

$$= F^2 B_a \otimes B_a + F(1 - F) \text{sym} B_a \otimes B_a + (1 - F)^2 B_b \otimes B_b,$$

(51)
yields different contributions in the $B_a$ and the $B_b$ directions. These are independent of the particular choice of $B_n$ according to the construction of the vertex model. A rotational symmetric incremental plastic stiffness tensor

$$\bar{K}_p^{\text{sym}} = F^2 B_a \otimes B_a + F(1 - F) \text{sym} B_a \otimes \left(\sum B_b\right)$$

$$+ (1 - F)^2 \sum B_b \otimes B_b,$$

(52)
can be obtained using all 5 symmetric tensorial direction perpendicular to $B_a$, called $B_{ab}$. Thus the matrix representation in the tensorial base $\{B_a, B_{ab}\}$ of $\bar{K}_p$ is changed according to

$$\begin{bmatrix}
a & b & 0 & 0 & 0 & 0 \\
b & c & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
a & b & b & b & b & b \\
b & c & 0 & 0 & 0 & 0 \\
b & 0 & c & 0 & 0 & 0 \\
b & 0 & 0 & c & 0 & 0 \\
b & 0 & 0 & 0 & c & 0 \\
b & 0 & 0 & 0 & 0 & c \\
\end{bmatrix},$$

(53)

allowing for an arbitrary secondary path with a constant incremental stiffness.

For the $a$-dependence we choose the minimizing result for $a = 0$ (46).

Table 3

Iterative solution of the buckling problem.

1. Fix a value of $a$ and compute the hardening modulus $h(z)$, the rotationally symmetric incremental stiffness $K_e$ and the equivalent stress $g(z)$.
2. Estimate the initial thickness $T_0$.
3. Create the FEM model for the given geometry and material parameters and compute the buckling load factor $\lambda$. Being a line load, it determines the uniaxial stress $\sigma_x = \frac{F}{A}$.
4. From the analytical result, compute the next iterate for the thickness, $T_{i+1} = T_i \left(\frac{K_e}{K_g}\right)$.
5. Until convergence of $T$ (and thus $a$) repeat from 2.

Iterating over geometrical parameters. For a fixed length $L$ and arm width $B$ of the cruciform, a simple fix point iteration for the thickness $T$ is used (see Table 3). The update equation is based on the analytical buckling equation (38). It is found that a single iteration step is sufficient to obtain four significant figures of the thickness.

To facilitate the iterative approach, a PYTHON script has been implemented that computes the incremental stiffness matrix, uses the preprocessor ABAQUS CAE in the model building process, submits the created jobs to ABAQUS ANALYSIS, and displays the results. The reading of the buckling load cannot be automated in the same fashion since it is not included in the machine readable ABAQUS output database. Thus it has to be read from the displayed result and typed in the iteration script manually by the user.

Typical buckling modes for the vertex model and the according elastic behaviour can be found in the Figs. 11 and 12. The differences

Fig. 11. First buckling modes for the vertex model and for elastic behaviour.
between elastic and inelastic buckling shapes are marginal. Obviously, the shape is dominated by the shell kinematics and the support of the different edges and not by the material equations.

Both the first and the second mode have a torsion angle varying with the length coordinate. In contrast, analytical torsion theory would predict a constant twist per unit length.

The resulting buckling loads have been converted to stresses and with the definition of the buckling reduction factor

\[ \eta = \frac{\tau_{\text{vertex}}}{\tau_{\text{elastic}}} \]

The results have been added to the diagram in Fig. 9, reproducing the analytical curve at a little too high level.

Similar results have been obtained by Papadopoulos and Lu (1998). However, their computational approach is based on Naghdi’s version of finite plasticity with a symmetric plastic state variable. Moreover, a return-mapping algorithm is used that artificially introduces a vertex into smooth models of plasticity if only the time step is large enough.

6. Conclusion

As an extension of the classical buckling equation by Hutchinson and Budiansky (1976), the reduced modulus in the buckling equation of a cruciform column has been obtained as a weighted average of the incremental elastic–plastic moduli over the cross-section of the cruciform’s flanges. It has been shown that the loading parameter \( \alpha \) characterizing the amount of deviation from proportional straining on the primary paths is non-uniform in the flanges’ cross-section. Accordingly, the average is non-trivial in presence of the vertex effect, while for uniform moduli (as in both the \( f_2 \) flow and deformation theory) the classical equation results.

The resulting buckling load reduction factor compared to the elastic behaviour found in experiments and by application of the
deformation theory (Gerard and Becker, 1957) can be reproduced with the vertex plasticity model proposed by Schurig and Bertram (submitted for publication).

In contrast to the \(J_2\) flow theory that by its assumptions predicts pure elastic behaviour for a shear component superimposed to plastic compression due to the restriction of the plastic flow by the normality rule, and also contrasting the \(J_2\) deformation theory, that always assumes total loading, the vertex plasticity model was developed to describe the plastic flow of a polycrystal after a strain path change. Accordingly, the activity of distinct potentially active plastic mechanisms has not been set by a-priori assumptions of the constitutive model, but obtained using transition functions which have been identified from comparison with a numerical polycrystal model. The deviation from proportionality and the resulting moduli describe a partially unloading state. Therefore, the vertex effect and application of a vertex plasticity model are relevant for the problem. No imperfections or finite timesteps in a numerical scheme have been applied that both trigger plastic behaviour even for the \(J_2\) flow theory. The good results obtained in calculating the buckling load reduction factor using the deformation theory can be explained by the non-uniformity of the loading parameter and the the total-loading to total-unloading transition of the incremental elastic–plastic shear modulus, that attains its minimal value in most of the cross section for a sufficiently large twisting rate. Accordingly, the lower bound supplied by the deformation theory is a good approximation for this particular case.

The validity of the model extends to the secondary path as well, as long as the polycrystal response to a strain path change can be described by the vertex effect alone and no directional hardening or texture evolution takes place. By the energy criterion of path stability, the torsion-dominant buckling mode has been shown to be the stable post-critical path.

It is an advantage of the vertex plasticity model over using the flow theory without vertex effect, that the parameters that govern both the buckling load and the post-critical path are physically determined by the transition from total loading to total unloading of the plasticity model. There is no need for finite timesteps or assumed geometric imperfections of finite amplitude that introduce additional phenomenological parameters that must be identified from experiments. The latter is also an advantage over the combination of the deformation theory for prediction of the buckling load, and the flow theory with finite imperfections to predict the secondary path, in spite of the accurate results that are possible by this combination.

Acknowledgement

The research has been carried out at the University of Magdeburg with partial support by DFG (Deutsche Forschungsgemeinschaft), through the graduate school Micro–Macro Interactions in Structured Media and Particle Systems.

The comments received from the anonymous reviewers are gratefully acknowledged.

References

Schurig, M., Bertram, A., submitted for publication. A material model for the vertex effect in polycrystal plasticity based on a modified plastic potential. IJS, submitted for publication.