J Elasticity (2007) 86:235–243
DOI 10.1007/s10659-006-9091-z

On the Rank 1 Convexity of Stored Energy Functions of Physically Linear Stress-Strain Relations

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Received: 1 June 2005 / Accepted: 27 September 2006 /
Published online: 16 November 2006

Abstract The rank 1 convexity of stored energy functions corresponding to isotropic and physically linear elastic constitutive relations formulated in terms of generalized stress and strain measures [Hill, R.: J. Mech. Phys. Solids **16**, 229–242 (1968)] is analyzed. This class of elastic materials contains as special cases the stress-strain relationships based on Seth strain measures [Seth, B.: Generalized strain measure with application to physical problems. In: Reiner, M., Abir, D. (eds.) Second-order Effects in Elasticity, Plasticity, and Fluid Dynamics, pp. 162–172. Pergamon, Oxford, New York (1964)] such as the St.Venant–Kirchhoff law or the Hencky law. The stored energy function of such materials has the form

$$\tilde{W}(\boldsymbol{F}) = W(\alpha) := \frac{1}{2}\sum_{i=1}^{3} f(\alpha_i)^2 + \beta \sum_{1\le i<j\le 3} f(\alpha_i) f(\alpha_j),$$

where $f : (0,\infty) \to \mathbb{R}$ is a function satisfying $f(1) = 0$, $f'(1) = 1$, $\beta \in \mathbb{R}$, and α_1, α_2, α_3 are the singular values of the deformation gradient $\boldsymbol{F}$. Two general situations are determined under which $\tilde{W}$ is not rank 1 convex: (a) if (simultaneously) the Hessian of W at $\alpha = (1, 1, 1)$ is positive definite, $\beta \neq 0$, and f is strictly monotonic, and/or (b)

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if f is a Seth strain measure corresponding to any $m \in \mathbb{R}$. No hypotheses about the range of f are necessary.

Key words generalized linear elastic laws · generalized strain measures · rank 1 convexity

Mathematics Subject Classifications (2000) 74A20 · 74B05 · 74B20

1 Introduction

Hooke's law is the geometrically and physically linear relation between the true or Cauchy stress tensor and the infinitesimal strain tensor. It is one of the oldest and, mathematically and physically, best understood constitutive relations in continuum mechanics. The geometrically nonlinear but physically linear formulation of Hooke's law is not unique in its form. This is due to the fact that there are arbitrarily many generalized stress and strain measures [5] which, of course, coincide for small strains.

In this paper we consider the class of isotropic and physically linear but geometrically nonlinear elastic laws. Although under large strains a physically linear elastic model can give only a very rough description of the material behavior, the question of existence and uniqueness is of great importance. Statements of existence and uniqueness of solutions of boundary value problems require certain restrictions upon the elastic strain energy density. The theoretically well-motivated concept of quasiconvexity has the disadvantage that for specific functions it is difficult to be verified. Convexity and polyconvexity [1] are sufficient conditions for quasiconvexity [3, 6, 11]. But the restriction of convexity was realized as too strong, and nowadays polyconvexity is frequently required. Polyconvexity is essentially very close to quasiconvexity so that it can be required instead of quasiconvexity without any practical difference. The condition of polyconvexity became a powerful means to establish the existence of solutions of extremum problems of solids. Rank 1 convexity is a necessary condition for quasiconvexity and the difference of both is also relatively small. Furthermore, rank 1 convexity implies the Legendre–Hadamard condition which is also called condition of infinitesimal rank 1 convexity. In the paper we show that under quite general assumptions the class of isotropic and physically linear but geometrically nonlinear elastic laws fail to be rank 1 convex.

The outline of the paper is as follows. In Section 2 the generalized strains according to Seth [9] and Hill [5] are introduced. In order to make the paper self-contained, we summarize the formulae for the determination of the generalized stress tensor which is conjugate to a given generalized strain tensor. In Section 3 elastic and hyperelastic constitutive relations in terms of generalized stress and strain measures are considered. For the special case of a physically linear and isotropic behavior the stored energy function $\tilde{W}$ is introduced. In Section 4 two general situations are determined under which $\tilde{W}$ is not rank 1 convex: (a) if (simultaneously) the Hessian of W at $\alpha = (1, 1, 1)$ is positive definite, $\beta \neq 0$, and f is strictly monotonic, and/or (b) if f is a Seth strain measure. No hypotheses about the range of f are necessary.

Notation Throughout the text a direct tensor notation is preferred. A linear mapping of a 2nd-order tensor is written as $\boldsymbol{A} = \mathbb{C}[\boldsymbol{B}]$. The scalar product and the dyadic product are denoted by $\boldsymbol{A} \cdot \boldsymbol{B} = \mathrm{tr}(\boldsymbol{A}^{\mathsf{T}} \boldsymbol{B})$ and $\boldsymbol{A} \otimes \boldsymbol{B}$, respectively. The sets of proper orthogonal and symmetric tensors are denoted by $Orth^+$ and Sym. Inv^+ is the set of invertible tensors with positive determinant.

2 Generalized Strain Measures

The right Cauchy–Green tensor is given by $\boldsymbol{C} = \boldsymbol{F}^{\mathsf{T}} \boldsymbol{F}$, where $\boldsymbol{F}$ denotes the deformation gradient. The generalized strains or Hill strains $\boldsymbol{E}$ [5] are defined in terms of $\boldsymbol{C}$ by

$$\boldsymbol{E} = \sum_{i=1}^{\gamma} \boldsymbol{k}(\boldsymbol{C}) = \sum_{i=1}^{\gamma} k(c_i) \boldsymbol{P}_i, \tag{1}$$

where c_i and $\boldsymbol{P}_i$ denote the eigenvalues and eigenprojections of $\boldsymbol{C}$, respectively. γ is the number of distinct eigenvalues of $\boldsymbol{C}$. The function k is assumed to be twice continuously differentiable and monotonous with the properties

$$k(1) = 0, \qquad k'(1) = \frac{1}{2}, \qquad k' \neq 0. \tag{2}$$

A special class of generalized strains, the Seth strain measures [9], are defined by the function ($m \in \mathbb{R}$)

$$k(x) = \begin{cases} \dfrac{1}{2m}\left(x^m - 1\right) \mid m \neq 0 \\ \\ \dfrac{1}{2}\ln(x) \qquad\quad \mid m = 0. \end{cases} \tag{3}$$

All commonly used strain measures fall within this class, such as Green's strain ($m = 1$), Biot's strain ($m = 1/2$), Hencky's strain ($m = 0$), or Almansi's strain ($m = -1$). For small deformations $\boldsymbol{E}$ coincides with the infinitesimal strain tensor.

A generalized stress tensor $\boldsymbol{T}$ can be defined by the requirement that the inner product $\boldsymbol{T} \cdot \dot{\boldsymbol{E}}$ is equal to the production of internal energy (or stress power) per unit referential volume [5], which generally can be written as $\boldsymbol{S} \cdot \dot{\boldsymbol{C}}/2$. $\boldsymbol{S}$ is the 2nd Piola–Kirchhoff stress tensor. A dot denotes the material time derivative. From

$$\frac{1}{2}\boldsymbol{S} \cdot \dot{\boldsymbol{C}} = \boldsymbol{T} \cdot \dot{\boldsymbol{E}}, \tag{4}$$

one concludes

$$\boldsymbol{S} = 2\left(\frac{\partial \boldsymbol{k}(\boldsymbol{C})}{\partial \boldsymbol{C}}\right)^{\mathsf{T}} [\boldsymbol{T}], \qquad \boldsymbol{T} = \frac{1}{2}\left(\frac{\partial \boldsymbol{k}(\boldsymbol{C})}{\partial \boldsymbol{C}}\right)^{-\mathsf{T}} [\boldsymbol{S}]. \tag{5}$$

The transposition refers to the major symmetry of the 4th-order tensor $\partial \boldsymbol{k}(\boldsymbol{C})/\partial \boldsymbol{C}$.

Based on the formulae for the derivative of the eigenvalues and eigenprojections, the spectral decomposition of $\partial \boldsymbol{k}(\boldsymbol{C})/\partial \boldsymbol{C}$ can be determined [2, 11, 13]. For simplicity, the case of distinct eigenvalues of $\boldsymbol{C}$ is considered. The gradient of $\boldsymbol{k}$ is

$$\frac{\partial \boldsymbol{k}(\boldsymbol{C})}{\partial \boldsymbol{C}} = \sum_{1 \leq i \leq j \leq 3} g_{ij} \boldsymbol{G}_{ij} \otimes \boldsymbol{G}_{ij} \tag{6}$$

with

$$g_{ij} = \begin{cases} k'(c_i) & \mid i = j \\ \dfrac{k(c_i) - k(c_j)}{c_i - c_j} & \mid i \neq j \end{cases} \tag{7}$$

and

$$\boldsymbol{G}_{ij} = \begin{cases} \boldsymbol{c}_i \otimes \boldsymbol{c}_i & \mid i = j \\ \dfrac{\sqrt{2}}{2}\left(\boldsymbol{c}_i \otimes \boldsymbol{c}_j + \boldsymbol{c}_j \otimes \boldsymbol{c}_i\right) & \mid i \neq j, \end{cases} \tag{8}$$

where the eigenvectors of $\boldsymbol{C}$ are denoted by $\boldsymbol{c}_i$. The g_{ij} are eigenvalues and the $\boldsymbol{G}_{ij}$ are the eigentensors of $\partial \boldsymbol{k}(\boldsymbol{C})/\partial \boldsymbol{C}$.

By applying (6), (7) and (8) to (5), we obtain

$$\boldsymbol{S} = 2 \sum_{1 \leq i \leq j \leq 3} g_{ij} T_{ij} \boldsymbol{G}_{ij}, \tag{9}$$

where the T_{ij} are the components of $\boldsymbol{T}$ with respect to the tensor basis $\boldsymbol{G}_{ij}$. For the special case that $\boldsymbol{T}$ and $\boldsymbol{E}$ are generally coaxial, as it is the case in isotropic elasticity, the last equation simplifies to

$$\boldsymbol{S} = 2 \sum_{i=1}^{3} g_{ii} T_{ii} \boldsymbol{G}_{ii} = 2 \sum_{i=1}^{3} k'(c_i) t_i \boldsymbol{P}_i, \qquad \boldsymbol{T} = t_i \boldsymbol{P}_i. \tag{10}$$

If the Seth strain measures are considered, the g_{ij} are

$$g_{ij} = \begin{cases} \dfrac{1}{2} c_i^{m-1} & \mid i = j \\ \dfrac{1}{2m} \dfrac{c_i^m - c_j^m}{c_i - c_j} & \mid i \neq j. \end{cases} \tag{11}$$

Equations (10) and (11) show that a coaxiality of $\boldsymbol{T}$ and $\boldsymbol{E}$ implies the following relation

$$\boldsymbol{S} = \sum_{i=1}^{3} c_i^{m-1} t_i \boldsymbol{P}_i = \boldsymbol{T}\boldsymbol{C}^{m-1}. \tag{12}$$

The generalized stresses are unique also for repeated eigenvalues since

$$\lim_{c_j \to c_i} \frac{k(c_i) - k(c_j)}{c_i - c_j} = k'(c_i). \tag{13}$$

From (7), (9), and (13) it is clear that for infinitesimal deformations the generalized stresses coincide with the Cauchy stress.

3 Physically Linear and Isotropic Elastic Stress-Strain Relations

For an elastic material the stresses are a function of the current deformation [10]. The equations

$$\boldsymbol{\sigma} = \boldsymbol{p}(\boldsymbol{F}), \qquad \boldsymbol{T} = \boldsymbol{q}(\boldsymbol{E}) \tag{14}$$

are general constitutive equations of an elastic material, where $\boldsymbol{\sigma}$ denotes the Cauchy stress tensor. Since $\boldsymbol{T}$ and $\boldsymbol{E}$ are independent of the observer, $(14)_2$ represents a reduced form of an elastic law. If the constitutive equations can be represented by a gradient of a stored energy function such that

$$\boldsymbol{\sigma} = \det \boldsymbol{F}^{-1} \frac{\partial \bar{W}(\boldsymbol{F})}{\partial \boldsymbol{F}} \boldsymbol{F}^{\mathsf{T}}, \qquad \boldsymbol{T} = \frac{\partial \hat{W}(\boldsymbol{E})}{\partial \boldsymbol{E}} \tag{15}$$

holds, then the elastic behavior is called hyperelastic.

The material symmetry of elastic solids can be defined by the symmetry group S with respect to the undistorted state. The elements of S satisfy

$$\boldsymbol{p}(\boldsymbol{F}) = \boldsymbol{p}(\boldsymbol{F}\boldsymbol{Q}^{\mathsf{T}}) \qquad \forall \boldsymbol{Q} \in S \subseteq Orth^{+},\ \forall \boldsymbol{F} \in Inv^{+} \tag{16}$$

or equivalently

$$\boldsymbol{Q}\boldsymbol{q}(\boldsymbol{E})\boldsymbol{Q}^{\mathsf{T}} = \boldsymbol{q}(\boldsymbol{Q}\boldsymbol{E}\boldsymbol{Q}^{\mathsf{T}}) \qquad \forall \boldsymbol{Q} \in S \subseteq Orth^{+},\ \forall \boldsymbol{E} \in Sym. \tag{17}$$

If $S = Orth^{+}$, then the elastic behavior is isotropic. The symmetry of a hyperelastic law is specified by

$$W(\boldsymbol{F}) = W(\boldsymbol{F}\boldsymbol{Q}^{\mathsf{T}}) \qquad \forall \boldsymbol{Q} \in S \subseteq Orth^{+},\ \forall \boldsymbol{F} \in Inv^{+} \tag{18}$$

and

$$W(\boldsymbol{E}) = W(\boldsymbol{Q}\boldsymbol{E}\boldsymbol{Q}^{\mathsf{T}}) \qquad \forall \boldsymbol{Q} \in S \subseteq Orth^{+},\ \forall \boldsymbol{E} \in Sym. \tag{19}$$

For physically linear and isotropic elastic laws, the function $\boldsymbol{q}(\boldsymbol{E})$ and the stored energy function $\hat{W}(\boldsymbol{E})$ are given by

$$\boldsymbol{T} = \boldsymbol{q}(\boldsymbol{E}) = \lambda \mathrm{tr}(\boldsymbol{E}) + 2\mu \boldsymbol{E} \tag{20}$$

and

$$\hat{W}(\boldsymbol{E}) = \frac{1}{2}(\lambda + 2\mu) \sum_{i=1}^{3} E_i^2 + \lambda \sum_{1 \le i < j \le 3} E_i E_j, \tag{21}$$

where the $E_i = k(c_i)$ are the eigenvalues of $\boldsymbol{E}$. The stored energy function $\bar{W}(\boldsymbol{F})$ is

$$\bar{W}(\boldsymbol{F}) = \frac{1}{2}(\lambda + 2\mu) \sum_{i=1}^{3} f(\alpha_i)^2 + \lambda \sum_{1 \le i < j \le 3} f(\alpha_i) f(\alpha_j), \tag{22}$$

where α_i are the singular values of $\boldsymbol{F}$. Due to $\alpha_i = \sqrt{c_i}$ we have $f(\alpha_i) = k(\alpha_i^2)$ and therefore from (2) one concludes that

$$f(1) = 0, \qquad f'(1) = 1, \qquad f' \neq 0. \tag{23}$$

In the following we use the normalized stored energy function

$$\tilde{W}(\boldsymbol{F}) = W(\alpha) = \frac{1}{2}\sum_{i=1}^{3} f(\alpha_i)^2 + \beta \sum_{1\le i<j\le 3} f(\alpha_i) f(\alpha_j) \tag{24}$$

which is obtained by dividing (22) by $\lambda + 2\mu$.

4 Statement of the Results

Consider an isotropic material with the strain energy function $\boldsymbol{F} \mapsto \tilde{W}(\boldsymbol{F})$ defined for all $\boldsymbol{F}$ with det $\boldsymbol{F} > 0$. We have

$$\tilde{W}(\boldsymbol{F}) = W(\alpha), \tag{25}$$

where $W : (0, \infty)^3 \to \mathbb{R}$ is a symmetric function and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ are the singular values of $\boldsymbol{F}$. The partial derivatives of W with respect to the singular values α_i are denoted by W_i. The following are necessary conditions for $\tilde{W}$ to be rank 1 convex:

(A) Baker–Ericksen inequalities:

$$(\alpha_i - \alpha_j)(\alpha_i W_i - \alpha_j W_j) \ge 0 \tag{26}$$

for every $\alpha \in (0, \infty)^3$, $1 \le i, j \le 3$; *equivalently*, [12], if $\alpha, \bar{\alpha} \in (0, \infty)^3$ satisfy $\alpha_1 \ge \alpha_2 \ge \alpha_3, \bar{\alpha}_1 \ge \bar{\alpha}_2 \ge \bar{\alpha}_3$, and

$$\alpha_1 \ge \bar{\alpha}_1, \qquad \alpha_1\alpha_2 \ge \bar{\alpha}_1\bar{\alpha}_2, \qquad \alpha_1\alpha_2\alpha_3 \ge \bar{\alpha}_1\bar{\alpha}_2\bar{\alpha}_3, \tag{27}$$

then

$$W(\alpha) \ge W(\bar{\alpha}); \tag{28}$$

(B) W is separately convex [4], i.e., for each $i = 1, 2, 3$, the function $\alpha_i \mapsto W(\alpha)$ is convex provided α_k, $k \ne i$ are fixed.

Suppose that W is of the form (24) for each $\alpha \in (0, \infty)^3$, where $f : (0, \infty) \to \mathbb{R}$ is a function satisfying $f(1) = 0$, $f'(1) = 1$, and $\beta \in \mathbb{R}$. A linearization at $\alpha = (1, 1, 1)$ gives an isotropic material with

$$\boldsymbol{\sigma} = \lambda \text{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}, \tag{29}$$

where

$$\lambda = \beta, \qquad 2\mu = 1 - \beta \tag{30}$$

and $\boldsymbol{\varepsilon}$ is the infinitesimal strain tensor. Note that here $\boldsymbol{\sigma}$ represents the Cauchy stress tensor normalized by $\beta = \lambda/(\lambda + 2\mu)$. The Legendre–Hadamard condition at diag(1, 1, 1) requires $\lambda + 2\mu \ge 0$, $\mu \ge 0$ which is equivalent to

$$\beta \le 1. \tag{31}$$

The Hessian matrix of W at $\alpha = (1, 1, 1)$ is

$$\begin{bmatrix} 1 & \beta & \beta \\ \beta & 1 & \beta \\ \beta & \beta & 1 \end{bmatrix} \tag{32}$$

and this matrix is positive definite if and only if

$$-\frac{1}{2} < \beta < 1. \tag{33}$$

The bulk modulus is positive if and only if $3\lambda + 2\mu > 0$ which is equivalent to

$$\beta > -\frac{1}{2}. \tag{34}$$

Proposition 1 If f is strictly monotonic, $\beta \neq 0$, and the bulk modulus is positive then W never satisfies both (A) and (B). In particular, $\tilde{W}$ is not rank 1 convex.

Proof Suppose that W satisfies (A) and (B) and derive a contradiction. The separate convexity requires

$$f''(\alpha_1)(f(\alpha_1) + \beta(f(\alpha_2) + f(\alpha_3))) + f'(\alpha_1)^2 \geq 0 \tag{35}$$

for each $\alpha \in (0, \infty)^3$. For $\alpha_1 = \alpha_2 = 1$ this gives in particular that $f^2(\cdot)$ is convex and thus the limits

$$L := \lim_{s \to 0} f^2(s), \qquad M := \lim_{s \to \infty} f^2(s) \tag{36}$$

exist and $0 < L \leq \infty$, $0 < M \leq \infty$ where the positivity of L, M follows from the strict monotonicity of f and the fact that $f(1) = 0$. Thus at least one of the following possibilities occurs: (a) $0 < L < \infty$, (b) $0 < M < \infty$, (c) $L = M = \infty$. Let (a) hold. Let $1 < \xi < 1$ and let $\alpha = (1, 1, \xi)$, $\bar{\alpha} = (1, \sqrt{\xi}, \sqrt{\xi})$. The inequality (28) gives

$$\frac{1}{2} f^2(\xi) \geq f^2\left(\sqrt{\xi}\right) + \beta f^2\left(\sqrt{\xi}\right). \tag{37}$$

Letting $\xi \to 0$ and dividing the resulting inequality by $L > 0$ one obtains

$$\beta \leq -\frac{1}{2}, \tag{38}$$

which contradicts (34). Let (b) hold. Let $1 < \xi < \infty$ and let $\alpha = (\xi, 1, 1)$, $\bar{\alpha} = (\sqrt{\xi}, \sqrt{\xi}, 1)$. The inequality (28) gives (37). Letting $\xi \to \infty$ and dividing by M we again obtain (38), again in a contradiction with (34). Let (c) hold. Then the monotonicity of f implies that the range of f is $\mathbb{R}$. Setting $\alpha_3 = 1$ in (35) and letting $\alpha_2 \to 0$ and $\alpha_2 \to \infty$, we obtain $\beta f''(\alpha_1) = 0$. As $\beta \neq 0$, we have $f''(s) = 0$ for all $s > 0$ and hence $f(s) = s - 1$ for each $s > 0$. The latter is inconsistent with $L = M = \infty$. □

Proposition 2 Let $m \in \mathbb{R}$ be fixed and let

$$f(s) = \begin{cases} \dfrac{1}{m}\left(s^m - 1\right) & \text{if } m \neq 0 \\ \ln(s) & \text{if } m = 0 \end{cases} \tag{39}$$

for each $s > 0$. If $m \neq 0$ then W violates (A) while if $m = 0$ the W violates (B). In particular, if $m \in \mathbb{R}$ then $\tilde{W}$ is not rank 1 convex.

Proof Let first $m \neq 0$. The Baker–Ericksen inequality $(\alpha_i - \alpha_j)(\alpha_i W_i - \alpha_j W_j) \geq 0$ gives

$$\frac{1}{m}(\alpha_1^m - \alpha_2^m)(\alpha_1 - \alpha_2)(\alpha_1^m + \alpha_2^m - 1 - 2\beta + \beta\alpha_3^m) \geq 0; \tag{40}$$

as $(\alpha_1^m - \alpha_2^m)(\alpha_1 - \alpha_2)/m > 0$, we have

$$\alpha_1^m + \alpha_2^m - 1 - 2\beta + \beta\alpha_3^m \geq 0. \tag{41}$$

Letting $\alpha_1^m \to 0, \alpha_2^m \to 0, \alpha_3^m \to 0$, we obtain

$$1 + 2\beta \leq 0; \tag{42}$$

setting $\alpha_1 = \alpha_2 = 1$ and letting $\alpha_3^m \to \infty$ we obtain

$$\beta \geq 0. \tag{43}$$

Clearly, (42) and (43) are inconsistent. Thus the Baker–Ericksen inequalities are violated if $m \neq 0$. Finally if $m = 0$ then $s \mapsto \ln(s)^2$ must be convex. This gives $1 - \ln(s) \geq 0$ which is violated for large s. □

5 Summary

If an isotropic elastic stress–strain relationship is physically linearized in a Seth strain measure and the corresponding stiffness tensor is positive definite, then the corresponding strain energy density is not rank 1 convex. For the special case of a linearization in the Green–Lagrange strain tensor, i.e., $m = 1$, this has been shown already by [6] and [8]. For a linearization in Hencky's strain tensor, i.e., $m = 0$, this fact was noted by [7]. It has been shown in this paper that the aforementioned statement also holds for the more general class of Hill strains. Hence, if a polyconvex or a quasiconvex strain energy density is required in the isotropic hyperelastic case, then the stress-strain relationship must be physically non-linear in the generalized strain measure.

Acknowledgement The research of M. Šilhavý was supported by the grant of MIUR (Italy) "Variational theory of microstructure, semiconvexity, and complex materials." The support is gratefully acknowledged.

References

1. Ball, J.: Convexity conditions and existence theorems in nonlinear elasticity. Arch. Rational Mech. Anal. **63**, 337–403 (1977)
2. Carlson, D.E., Hoger A.: The derivative of a tensor-valued function of a tensor. Quart. Appl. Math. **44**, 409–423 (1986)
3. Ciarlet, P.: Mathematical Elasticity: Three-dimensional Elasticity, vol. I. Elsevier, Amsterdam (1988)
4. Dacorogna, B.: Direct Methods in the Calculus of Variations. Springer, Berlin Heidelberg New York (1989)
5. Hill, R.: On constitutive inequalities for simple materials. J. Mech. Phys. Solids **16**, 229–242 (1968)
6. Krawietz, A.: Materialtheorie. Springer, Berlin Heidelberg New York (1986)
7. Neff, P.: Mathematische Analyse Multiplikativer Elastoplastizität. Berichte aus der Mathematik. Shaker, Verlag (2000)

8. Raoult, A.: Non-polyconvexity of the stored energy function of a Saint Venant–Kirchhoff material. Appl. Math. **6**, 417–419 (1986)
9. Seth, B.: Generalized strain measure with application to physical problems. In: Reiner, M., Abir, D. (eds.) Second-order Effects in Elasticity, Plasticity, and Fluid Dynamics, pp. 162–172. Pergamon, Oxford, New York (1964)
10. Truesdell, C., Noll, W.: The Non-linear Field Theories of Mechanics, vol. III/3 of Encyclopedia of Physics. Springer, Berlin Heidelberg New York (1965)
11. Šilhavý, M.: The Mechanics and Thermodynamics of Continuous Media. Springer, Berlin Heidelberg New York (1997)
12. Šilhavý, M.: Monotonicity of rotationally invariant convex and rank 1 convex functions. Proc. Roy. Soc. Edinburgh **13A**, 419–435 (2002)
13. Xiao, H., Bruhns, O., Meyers, A.: Strain rates and material spins. J. Elasticity **52**, 1–41 (1998)