Chapter 6
Formulations of Strain Gradient Plasticity

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Abstract In the literature, different proposals for a strain gradient plasticity theory exist. So there is still a debate on the formulation of strain gradient plasticity models used for predicting size effects in the plastic deformation of materials. Three such formulations from the literature are discussed in this work. The pros and the cons are pointed out at the light of the original solution of a boundary value problem that considers the shear deformation of a periodic laminate microstructure.

Key words: Strain gradient plasticity. Continuum thermodynamics. Laminates. Constrained plasticity.

6.1 Introduction

The objective of this work is to present three main formulations of strain gradient plasticity that are available in the literature and to illustrate the pros and the cons of these approaches by means of a specific example for which an analytical solution is derived. The targeted model is one of the most simple strain gradient plasticity model which serves as a paradigm for most available strain gradient theories, namely the well–known Aifantis model [1]. For that purpose we start from an initial plasticity model for which the set of degrees of freedom and of the state variables are defined as follows:

$$DOF0 = \{\underline{u}\} \quad STATE0 = \{\underset{\sim}{\varepsilon}, \quad \alpha\}$$

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The displacement degrees of freedom are denoted by the vector $\underline{u}$ from which the linear strain tensor $\underset{\sim}{\varepsilon}$ is derived. The hardening/softening properties of materials are accounted for by means of internal variables, α, that can be tensors of any rank[1]. Examples for such internal variables in the context of isotropic plasticity are

$$\alpha = p,\, \underset{\sim}{\varepsilon}^p, \underset{\sim}{X} \ldots$$

where $\underset{\sim}{\varepsilon}^p$ is the plastic strain tensor, p is the cumulative plastic strain, that will be used for isotropic hardening, and $\underset{\sim}{X}$, the kinematic hardening variable [2, 3]. The continuum thermomechanics framework with internal variables has been settled in [4, 5, 6]. Internal variables are computed by integrating the evolution equations that are time differential equations. It has already been recognized that these evolution equations may well result from approximations of more general partial differential equations where the spatial derivatives are neglected due to the rapid local variations [7]. The objective of gradient theories is therefore to restore the status of internal degree of freedom to internal variables. Depending on the order of the partial differential equations, additional boundary conditions are usually necessary to solve boundary value problems. In the following, we call

- *internal variables*: state variables, the evolution of which is controlled by time differential equations;
- *internal degrees of freedom*: state variables the evolution of which is controlled by time and space partial differential equations, without need for additional boundary conditions;
- *degrees of freedom*: variables (not necessarily state variables) controlled by a space and time partial differential equations, the resolution of which requires additional boundary conditions to be specified.

The question arises how to enlarge the space of state variables to the gradient of α–variables, so as to introduce characteristic lengths in the continuum modeling:

$$STATE = \{\underset{\sim}{\varepsilon}, \quad \alpha, \quad \nabla\alpha\}$$

Such a gradient term enters in particular Aifantis isotropic model that postulates the following evolution of the equivalent stress measure under plastic loading:

$$\sigma_{eq} = R_0 + Hp - c\nabla^2 p \tag{6.1}$$

where R_0 is the initial yield strength, H is the classical hardening modulus and c denotes the square of a characteristic length. Various attempts have been proposed in order to derive the Laplace term introduced in the yield function from a consistent thermomechanical setting. The first proposal in [8] will be recalled in Sect. 6.2.2. It is based on the introduction of an extra–entropy flux. In contrast, other authors have tried to circumvent the introduction of extra–entropy flux or extra energy terms by setting specific boundary conditions associated to the higher order partial differential equations, as shown in Sect. 6.2.1. An alternative approach is to formulate an

[1] In the present contribution, the variable α is treated as a scalar, for the sake of simplicity.

extended principle of virtual power, as initially proposed by [9] for damage variables. This amounts to raising the status of internal variable to additional degrees of freedom. This is the subject of Sect. 6.3 where the original principle of virtual power [10] is extended to gradient variables in the spirit of [11]. This track has been followed in the last ten years in the following works [12, 13, 14, 15].

Finally, a boundary value problem on a periodic two–phase laminate microstructure under shear loading conditions is solved in order to illustrate the new boundary or interface conditions and determine the variables which are discontinuous across the interface. This example has been originally handled for Cosserat and micromorphic single crystals in [16, 17], but it is solved here for the first time for the Aifantis model, so that comparisons will be drawn with other generalized continuum theories.

Throughout the work, for the sake of conciseness, the temperature θ is assumed to be uniform and constant.

6.2 Derivation Based on the Exploitation of the Entropy Principle

In this section, the energy principle is assumed to hold in its usual local form

$$\dot{e} = \mathscr{P}^{(i)}, \quad \text{with} \quad \mathscr{P}^{(i)} = \underset{\sim}{\sigma} : \underset{\sim}{\dot{\varepsilon}} \tag{6.2}$$

where e is the internal energy density and $\mathscr{P}^{(i)}$ the usual power density of internal forces. The Helmholtz free energy density, $\psi = e - \eta\theta$, is assumed to depend on the already defined set $STATE$ and we give the following names to the partial derivatives with respect to α and its gradient:

$$\psi = e - \theta\eta\,, \quad a = -\frac{\partial\psi}{\partial\alpha}, \quad \underline{b} = -\frac{\partial\psi}{\partial\nabla\alpha} \tag{6.3}$$

where η is the entropy density function.

6.2.1 Vanishing Generalized Tractions

The entropy principle is now postulated first in its global form on the material domain V

$$\int_V \theta\dot{\eta}\, dV \geq 0$$

and converted into the Clausius–Duhem inequality

$$\int_V \left(\mathscr{P}^{(i)} - \dot{\psi}\right) dV \geq 0 \tag{6.4}$$

$$\int_V \left[\left(\underset{\sim}{\sigma} - \frac{\partial \psi}{\partial \underset{\sim}{\varepsilon}} \right) : \dot{\underset{\sim}{\varepsilon}} + a\dot{\alpha} + \underline{b} \cdot \nabla \dot{\alpha} \right] dV \geq 0 \tag{6.5}$$

in the absence of extra–entropy flux. The global Clausius–Duhem inequality is transformed in the following way:

$$\int_V \left[\left(\underset{\sim}{\sigma} - \frac{\partial \psi}{\partial \underset{\sim}{\varepsilon}} \right) : \dot{\underset{\sim}{\varepsilon}} + a\dot{\alpha} - \dot{\alpha}\mathrm{div}\,\underline{b} + \mathrm{div}\,(\dot{\alpha}\underline{b}) \right] dV \geq 0 \tag{6.6}$$

$$\int_V \left[\left(\underset{\sim}{\sigma} - \frac{\partial \psi}{\partial \underset{\sim}{\varepsilon}} \right) : \dot{\underset{\sim}{\varepsilon}} + (a - \mathrm{div}\,\underline{b})\dot{\alpha} \right] dV + \int_{\partial V} \dot{\alpha}\underline{b} \cdot \underline{n}\, dS \geq 0 \tag{6.7}$$

It is temptating to assume at this stage that the flux of $\underline{b}$ vanishes at the boundary of the domain V

$$\underline{b} \cdot \underline{n} = 0, \quad \forall \underline{x} \in \partial V \tag{6.8}$$

This condition corresponds to a Neumann extra–boundary condition for the partial differential equation for α. It follows that the residual dissipation takes the following canonical form involving the rate of the α–variable and the associated thermodynamical force and dissipation potential

$$\int_V \mathscr{A}\,\dot{\alpha}\, dV \geq 0, \quad \mathscr{A} := a - \mathrm{div}\,\underline{b} \tag{6.9}$$

in addition to the state law $\underset{\sim}{\sigma} = \partial \psi / \partial \underset{\sim}{\varepsilon}$.
Positivity of dissipation can then be ensured by the choice of a convex dissipation potential Ω providing the evolution equation for α:

$$\dot{\alpha} = \frac{\partial \Omega}{\partial \mathscr{A}} \tag{6.10}$$

This condition of vanishing flux at a boundary is discussed in [18] in the context of generalized standard gradient models.

On which domain V of the material body should the previous reasoning be applied? In principle, the thermodynamical statements are to be applied to each subdomain of the body. But it is hard to believe that the condition of vanishing generalized traction will be applied to the boundary of any subdomain. This point will be checked in the analytical example of Sect. 6.4. In the literature, the condition is usually limited to the outer boundary of the considered body (so–called *insulation condition* in [19]), or at the boundary of the part of the body which undergoes plastic loading. The latter applies to the finite element implementation of such gradient models, as proposed in [20].

6.2.2 Extra–entropy Flux

In general, according to the thermodynamics of irreversible processes [6], an extra–entropy flux in the entropy inequality cannot be excluded. It is introduced in the form of the vector field $\underline{k}$ in the local form of the entropy imbalance:

$$\dot{\eta} + \mathrm{div}\underline{k} \geq 0 \tag{6.11}$$

In the isothermal case, the Clausius–Duhem inequality then takes the form:

$$\mathscr{P}^{(i)} - \dot{\psi} + \mathrm{div}\, \theta \underline{k} \geq 0 \tag{6.12}$$

The exploitation of Clausius–Duhem inequality continues as follows:

$$(\underset{\sim}{\sigma} - \frac{\partial \psi}{\partial \underset{\sim}{\varepsilon}}) : \dot{\underset{\sim}{\varepsilon}} + a\dot{\alpha} + \underline{b} \cdot \nabla \dot{\alpha} + \mathrm{div}\, \theta \underline{k} \geq 0 \tag{6.13}$$

$$(\underset{\sim}{\sigma} - \frac{\partial \psi}{\partial \underset{\sim}{\varepsilon}}) : \dot{\underset{\sim}{\varepsilon}} + (a - \mathrm{div}\underline{b})\dot{\alpha} + \mathrm{div}\,(\dot{\alpha}\underline{b} + \theta \underline{k}) \geq 0 \tag{6.14}$$

At this point, the following astute choice of the extra–entropy flux is proposed in [8]

$$\underset{\sim}{\sigma} = \frac{\partial \psi}{\partial \underset{\sim}{\varepsilon}}, \quad \underline{k} := -\frac{\dot{\alpha}}{\theta}\underline{b} \tag{6.15}$$

With this choice, the residual dissipation reduces to the same form as (6.9), so that again a dissipation potential $\Omega(\mathscr{A})$ can be introduced, thus setting the framework of standard generalized gradient models. The difference compared to the previous approach is that no restriction arises in the derivation concerning the additional boundary condition to solve (6.1). As a result, the flux $\underline{b} \cdot \underline{n}$ can take any needed values at boundaries and interfaces. The approach provides no indication nor restrictions on the necessary boundary conditions.

6.3 Derivation Based on the Modification of the Energy Principle

An alternative to the previous approaches is to consider that the introduction of mechanical gradient effects must be accompanied by a modification of the power of internal forces which enters the principle of virtual power. When higher order gradients of the displacement field exist like in Mindlin's second gradient theory [11, 21, 22] or gradients of additional degrees of freedom, like in Eringen's micromorphic model [23], the power of internal variable is extended to include a power induced by the higher order gradients or the gradients of additional degrees of freedom. Let us consider for instance Mindlin's second gradient model which incorporates the effect of the strain gradient $\nabla \underset{\sim}{\varepsilon}$. The stress conjugate of the strain rate gradient in the power of internal forces is the third rank double stress tensor. If the strain is decomposed into elastic and plastic contributions,

$$\underset{\sim}{\varepsilon} = \underset{\sim}{\varepsilon}^e + \underset{\sim}{\varepsilon}^p, \tag{6.16}$$

one may consider materials for which most of the gradient effects come from $\nabla \underset{\sim}{\varepsilon}^p$, so that the effect of $\nabla \underset{\sim}{\varepsilon}^e$ can be neglected. The latter term disappears but the triple contraction of the double stress and of the gradient of plastic strain remains. This suggests that when the gradient of α–variables is considered, one is entitled to introduce a corresponding internal power. This approach is presented in this section and has been followed in the references [9, 12, 15] for gradient of damage and plasticity models.

We introduce the enriched power density of internal forces and of contact forces

$$\mathscr{P}^{(i)} = \underset{\sim}{\sigma} : \dot{\underset{\sim}{\varepsilon}} + a\dot{p} + \underline{b} \cdot \nabla \dot{p}, \quad \mathscr{P}^{(c)} = \underline{t} \cdot \dot{\underline{u}} + a^c \dot{p} \tag{6.17}$$

where a and $\underline{b}$ are generalized stresses acting on the virtual field α and its gradient, respectively. The usual traction vector is $\underline{t}$ and a^c denotes the generalized traction. Such generalized stresses are called micro–forces in [14]. A generalized principle of virtual power is stated with respect to the virtual fields of displacements and the α–variable. The methodology originates from the works [10, 22] and was extended to generalized continua in [13, 24]. The application of this principle results in an additional balance equation, complementing the usual balance of momentum equation:

$$\operatorname{div} \underset{\sim}{\sigma} = 0, \quad a = \operatorname{div} \underline{b}, \quad \forall \underline{x} \in V \tag{6.18}$$

written here in the static case and in the absence of body forces. The corresponding equilibrium conditions at the boundaries are:

$$\underline{t} = \underset{\sim}{\sigma} \cdot \underline{n}, \quad a^c = \underline{b} \cdot \underline{n}, \quad \forall \underline{x} \in \partial V \tag{6.19}$$

An essential feature of the model is that the extended power of internal forces intervenes in the energy balance equation:

$$\dot{e} = \mathscr{P}^{(i)} \tag{6.20}$$

thus including the additional contributions of generalized stresses. This also holds for the entropy principle in its local form:

$$\mathscr{P}^{(i)} - \dot{\psi} \geq 0 \tag{6.21}$$

The Clausius–Duhem inequality then becomes:

$$(\underset{\sim}{\sigma} - \frac{\partial \psi}{\partial \underset{\sim}{\varepsilon}^e}) : \dot{\underset{\sim}{\varepsilon}}^e + (a - \frac{\partial \psi}{\partial \alpha})\dot{\alpha} + (\underline{b} - \frac{\partial \psi}{\partial \nabla \alpha}) \cdot \nabla \dot{\alpha} + \underset{\sim}{\sigma} : \dot{\underset{\sim}{\varepsilon}}^p \geq 0 \tag{6.22}$$

At this stage, we adopt the following state laws

$$\underset{\sim}{\sigma} = \frac{\partial \psi}{\partial \underset{\sim}{\varepsilon}^e}, \quad a = \frac{\partial \psi}{\partial \alpha} - R, \quad \underline{b} = \frac{\partial \psi}{\partial \nabla \alpha} \tag{6.23}$$

thus assuming that no dissipation is associated with the generalized stress $\underline{b}$. This is the most simple assumption that is sufficient for deriving Aifantis model, in particular. R is the dissipative part of generalized stress a.

At this point it is more convenient to specify the internal variable that is required to derive Aifantis model. We adopt: $\alpha \equiv p$, so that the considered internal variable is the cumulative plastic strain. The residual dissipation is then

$$\underset{\sim}{\sigma} : \dot{\underset{\sim}{\varepsilon}}^p - R\dot{p} \geq 0 \tag{6.24}$$

Let us choose a simple quadratic free energy potential

$$\psi(\underset{\sim}{\varepsilon}^e, p, \nabla p) = \frac{1}{2} \underset{\sim}{\varepsilon}^e : \underset{\approx}{C} : \underset{\sim}{\varepsilon}^e + \frac{1}{2} H p^2 + \frac{1}{2} c \nabla p \cdot \nabla p \tag{6.25}$$

from which the state laws are derived:

$$\underset{\sim}{\sigma} = \underset{\approx}{C} : \underset{\sim}{\varepsilon}^e, \quad R = Hp - a, \quad \underline{b} = c \nabla p \tag{6.26}$$

where $\underset{\approx}{C}$ is the four–rank tensor of the elastic moduli, H is the usual hardening modulus and c is an additional material parameter (unit MPa.mm^2). The yield function is taken as

$$f(\underset{\sim}{\sigma}, R) = \sigma_{eq} - R_0 - R \tag{6.27}$$

Under plastic loading, this gives

$$\sigma_{eq} = R_0 + R = R_0 + Hp - a = R_0 + Hp - \mathrm{div}\, \underline{b} = R_0 + Hp - c\nabla^2 p \tag{6.28}$$

so that Aifantis equation (6.1) is recovered. The plasticity flow and evolution rules are

$$\dot{\underset{\sim}{\varepsilon}}^p = \dot{\lambda} \frac{\partial f}{\partial \underset{\sim}{\sigma}}, \quad \dot{p} = -\dot{\lambda} \frac{\partial f}{\partial R} = \dot{\lambda} \tag{6.29}$$

with λ being the plastic multiplier. These equations are used in the next section to solve a specific boundary value problem.

6.4 Analysis of a Simple Boundary Value Problem for Laminate Microstructures

Laminate microstructures are prone to size effects especially in the case of metals for which the interfaces act as barriers for the dislocations. The material response then strongly depends on the layer thickness. This situation has been considered for Cosserat and micromorphic single crystals under single and double slip in [16, 17]. The laminate microstructure is considered here in the case of Aifantis isotropic model. It is a periodic arrangement of two phases including a purely elastic material and a plastic strain gradient layer. The unit cell corresponding to this arrangement is shown in Fig. 6.1. It is periodic along all three directions of the space. It must must be replicated in the three directions so as to obtain the complete multilayer material. The thickness of the hard elastic layer is h, whereas the thickness of the soft plastic strain gradient layer is s.

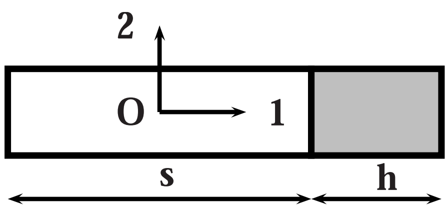

Fig. 6.1 Unit cell of a periodic two–phase laminate.

6.4.1 Position of the Problem

The unit cell of Fig. 6.1 is subjected to a mean simple shear $\overline{\gamma}$ in direction 1. The origin O of the coordinate system is the center of the soft phase. The displacement field is of the form

$$u_1 = \overline{\gamma}\, x_2, \quad u_2(x_1) = u(x_1), \quad u_3 = 0 \tag{6.30}$$

where $u(x_1)$ is a periodic function which describes the fluctuation from the homogeneous shear. This fluctuation is the main unknown of the boundary value problem. We compute the gradient of the displacement field and strain tensors:

$$[\nabla \underline{u}] = \begin{bmatrix} 0 & \overline{\gamma} & 0 \\ u_{,1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\underset{\sim}{\varepsilon}] = \begin{bmatrix} 0 & \frac{1}{2}(\overline{\gamma}+u_{,1}) & 0 \\ \frac{1}{2}(\overline{\gamma}+u_{,1}) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6.31}$$

where $u_{,1}$ denotes the derivative of the displacement u with respect to x_1. After Hooke's law, the only activated simple stress component is σ_{12}. Due to the balance of momentum equation and the continuity of the traction vector, this stress component is homogeneous throughout the laminate.

The elastic law in the elastic phase and the elastic–plastic response of the soft phase are then exploited in the next section to derive the partial differential equations for plastic strain and, finally, for the displacement fluctuation. The explicit solution is found after considering precise interface conditions regarding continuity of various variables.

Note that the solution is known for conventional plasticity, i.e. in the absence of strain gradient effect. The plastic strain is expected to be homogeneous in the soft phase for any loading $\bar{\gamma}$. Plastic strain therefore exhibits the usual jump at the interface. The introduction of higher order interface conditions, associated with strain gradient plasticity, will induce a non–homogeneous plasticity field.

6.4.2 Analytical Solution

Assuming plastic loading in the soft phase, the von Mises criterion is fulfilled:

$$\sqrt{3}|\sigma_{12}| = R_0 + Hp - cp_{,11} \tag{6.32}$$

Since the stress component σ_{12} is uniform, the previous equation can be differentiated with respect to x_1, which gives:

$$p_{,1} - \omega^{-2} p_{,111} = 0, \quad \omega^2 = \frac{H}{c} \tag{6.33}$$

The form of the plastic strain field therefore is

$$p = \alpha \cosh(\omega x_1) + \beta \tag{6.34}$$

where α and β are integration constants. In the elastic zone, the stress is given by

$$\sigma_{12} = \mu(\bar{\gamma} + u^h_{,1}) \quad \Longrightarrow \quad u^h_{,1} = C \tag{6.35}$$

where the uniformity of stress has been used again. An additional integration constant C must be determined. The exponent ^{h} has been added to indicate the displacement fluctuation inside the elastic phase. The arbitrary translation for u^h will be set to zero. The field u^s can be determined from the elasticity law in the soft phase:

$$\sigma_{12} = \mu(\bar{\gamma} u^s_{,1} - \sqrt{3}p) \tag{6.36}$$

An additional constant D arises from the integration of this equation, that remains to be determined.

The four unknown integration constants α, β, C, D will be determined from 4 conditions at the interface between both materials at $x_1 = \pm s/2$:

- Continuity of simple traction:

$$\sqrt{3}\mu(\bar{\gamma} + C) = R_0 + H\beta \tag{6.37}$$

- Continuity of displacement $u(x_1)$ at $s/2$:

$$u^s\left(\frac{s}{2}\right) = u^h\left(\frac{s}{2}\right) \tag{6.38}$$

$$u^h(x_1) = Cx_1, \quad u^s(x_1) = \left[\frac{R_0}{\mu\sqrt{3}} + (\frac{H}{\mu\sqrt{3}} + \sqrt{3})\beta - \overline{\gamma}\right] x_1 + \frac{\sqrt{3}\alpha}{\omega}\sinh(\omega x_1) + D \tag{6.39}$$

- Periodicity of displacement $u(x_1)$:

$$u^s\left(-\frac{s}{2}\right) = u^h\left(\frac{s}{2} + h\right) \tag{6.40}$$

- Continuity of plastic strain p at the interface $x_1 = \frac{s}{2}$

$$p\left(\frac{s}{2}\right) = 0 \tag{6.41}$$

$$\alpha\cosh\left(\omega\frac{s}{2}\right) + \beta = 0 \tag{6.42}$$

The last condition is necessary to close the system. Differentiability and hence continuity of plastic strain p is required in strain gradient plasticity theory. In the elastic phase, $p = 0$ so that p should also vanish at the interface.

The identification of the constants provides:

$$\beta = \frac{\left(\overline{\gamma} - \frac{R_0}{\mu\sqrt{3}}\right)(s+h)}{\frac{H}{\mu\sqrt{3}}(s+h) + \sqrt{3}s - \tanh(\omega\frac{s}{2})\frac{2\sqrt{3}}{\omega}} \tag{6.43}$$

$$\alpha = -\frac{\beta}{\cosh(\omega\frac{s}{2})} \tag{6.44}$$

$$C = \frac{R_0}{\mu\sqrt{3}} - \overline{\gamma} + \frac{H}{\sqrt{3}\mu}\beta \tag{6.45}$$

$$D = C\frac{s}{2} - \left[\frac{R_0}{\mu\sqrt{3}} + (\frac{H}{\mu\sqrt{3}} + \sqrt{3})\beta - \overline{\gamma}\right]\frac{s}{2} - \frac{\sqrt{3}\alpha}{\omega}\sinh\left(\omega\frac{s}{2}\right) \tag{6.46}$$

where homogeneous elasticity has been assumed for simplicity, with μ being the shear modulus of both phases.

As a result, we find that the double traction cannot vanish on the soft side of the interface, $x_1 = s^-/2$.

$$b_1(x_1) = c\alpha\sinh(\omega x_1), \quad b_1(\frac{s}{2}^-) = c\alpha\sinh\left(\omega\frac{s}{2}\right) \neq 0 \tag{6.47}$$

In the elastic phase, the generalized stress identically vanishes since no plastic strain occurs. It follows that the generalized traction b_1 exhibits a jump across the interface.

We illustrate the previous solution for a special choice of material parameters oriented towards plasticity of metals at the micron scale. The parameters used in the simulations are:

$s = 0.007$ mm, $h = 0.003$ mm, $\overline{\gamma} = 0.01$,
$\mu = 300$ GPa, $R_0 = 20$ MPa, $H = 10$ GPa, $c = 0.005$ MPa.mm^2.

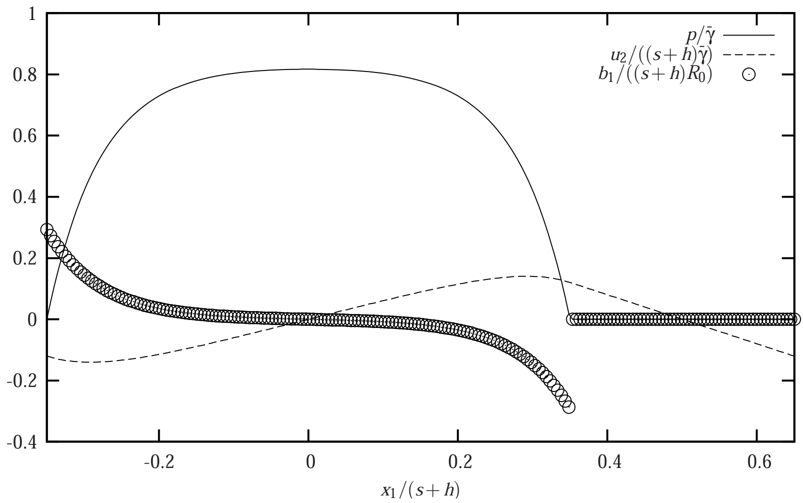

Fig. 6.2 Distributions of plastic strain, normalized displacement fluctuation and normalized generalized stress vector component in the unit cell of the laminate microstructure.

The distribution of plastic slip, of displacement and of generalized stress component b_1 are shown in Fig. 6.2. The plastic strain displays a typical *cosh* profile with boundary layer effects close to the interface, due to the continuity requirement. The displacement fluctuation is clearly periodic. The jump of the generalized traction at the interface is also visible.

6.5 Discussion

Three different formulations of strain gradient plasticity have been reported in this contribution. The first model is based on the assumption of vanishing general traction at the boundary of some domain and in particular, as advocated by several authors, at the interface between the elastic and plastic loading domain. The example of the laminate microstructure considered in Sect. 6.4 clearly shows that this assumption cannot be valid systematically. Indeed, if a condition of vanishing double traction is imposed on the interface $x_1 = s/2$ in the laminate microstructure, this amounts to prescribe vanishing of the plastic strain and its first derivative at the interface. Accordingly, the solution of the equation (6.33) yields $p = Cst$, which is the standard solution in classical plasticity.

The presented analytical example is compatible with the second formulation of strain gradient plasticity based on the introduction of an additional entropy flux.

The third approach based on the introduction of the extended power of internal forces has the advantage that it provides a variational formulation of the strain gradient plasticity boundary value problem in the form of a generalized principle of virtual power. This has a direct implication on the numerical treatment by means of the finite element method for instance. In the finite element implementation, the plastic strain is handled as an additional degree of freedom. The power of internal forces is discretized in space and the generalized stresses are computed from the constitutive equations (6.26). The plastic multiplier is computed by taking the enhanced hardening rule into account [24]. A Lagrange multiplier is then needed to ensure that the additional degree of freedom coincides with the time integrated cumulative plastic strain. The additional boundary condition arises naturally from the finite element formulation, the reaction to the nodal degrees of freedom being related to the generalized traction.

It seems that there is a real necessity for an energy cost associated with the development of the plastic strain gradient. Finite element simulations are presented in literature that include the plastic strain gradient, computed at the end of the increment, in the hardening rule. They do not consider generalized stresses nor associated additional boundary conditions. Such a procedure is known to lead to mesh–dependent results even in the hardening regime [25]. This pleads for the adoption of the third proposed approach to strain gradient plasticity.

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