



A rate-dependent endochronic approach to thermoplastic materials: temperature and filler volume fraction dependence

Thomas Kletschkowski^{a,*}, Uwe Schomburg^a, Albrecht Bertram^b

^a *Technische Mechanik, Universität der Bundeswehr Hamburg, Holstenhofweg 85, D-22043 Hamburg, Germany*

^b *Inst. f. Mechanik, Otto-von-Guericke-Universität Magdeburg, Universitätsplatz 2, D-39106 Magdeburg, Germany*

Received 12 December 2003; received in revised form 18 May 2004

Abstract

An endochronic viscoplastic approach, derived from the theory of finite viscoplasticity based on isomorphisms, is presented. The model allows to characterize viscoplastic material behaviour with equilibrium hysteresis using a rate-independent elastoplastic model with an endochronic flow rule and a logarithmic elastic law in parallel connection with a non-linear Maxwell model. The volumetric strains are assumed to be purely elastic. The proposed model describes the non-linear mechanical behaviour of thermoplastic materials such as filled polytetrafluoroethylene (PTFE). The temperature and filler volume fraction dependencies of the model parameters have been identified by a Levenberg–Marquardt algorithm. In contrast to previous suggestions, the proposed model is capable to describe both the non-linearities during strain controlled loading and the non-linearities during stress controlled unloading in a unified approach. The results of the numerical simulations are in fair agreement with the experimental data.

© 2004 Elsevier Ltd. All rights reserved.

Keywords: Endochronic theory of viscoplasticity; Thermoplastic materials; PTFE

1. Introduction

Due to their mechanical properties thermoplastic materials are widely used in numerous engineer-

ing applications. Typical examples are rotary shaft seals made of filled polytetrafluoroethylene (PTFE). The design engineer has to take into account the viscoplastic material behaviour of these thermoplastic compounds, which is radically different from that of elastomeric materials. In the past, constitutive models have been applied to simulate the mechanical response of thermoplastic materials, which had been successful only in particular load cases.

* Corresponding author. Tel.: +49 40 6541 2282; fax: +49 40 6541 2822.

E-mail address: kletsch@unibw-hamburg.de (T. Kletschkowski).

Short time processes had been usually calculated using an elastoplastic, (Sui, 1998; Pohl, 1999), or a viscoplastic approach with a strain rate dependent yield stress (Pohl, 1999). Viscoelastic approaches were applied to analyze long time processes (stress relaxation). To compute the stress relaxation curves after loading, Olbrich et al. (2000) used experimental data as initial values instead of the calculated data from the loading curves. Moreover, the numerical results of Sui (1998) and Pohl (1999) show that constitutive models with linear or quasilinear elastic ranges are not adequate to describe the non-linearities of thermoplastic materials during unloading. Furthermore, all these models are unable to separate the rate-dependent and the rate-independent phenomena by an additive decomposition of the total stress response of the material into a rate-dependent overstress and a rate-independent equilibrium stress.

Rate-dependent constitutive models without equilibrium hysteresis can be found in the calculations of Wüstenhagen (1995) and Lin (2003). The linear viscoelastic approach of Wüstenhagen (1995) is limited to very small strains. The more complex theory of finite linear viscoelasticity, used by Lin (2003), allows to consider physical and geometrical non-linearities during the computation. But the rate-independent plastic deformation (which can be clearly observed during tests) and the strength differential effect (SDE), a stress-strain-asymmetry in tension and compression, are completely ignored by the use of these purely viscoelastic approaches.

The viscoplastic model presented by Khan and Zhang (2001) is capable to simulate the creep behaviour of thermoplastic materials. But the numerical results of strain controlled simulations (stress relaxation) deviate noticeably from the corresponding experimental data.

In order to simulate the mechanical behaviour of thermoplastic materials successfully in a unified approach without neglecting important inelastic effects, which are clearly observed in the experiments and without switching from one constitutive model to an other, the endochronic viscoplastic material model, proposed by Kletschkowski et al. (2002), is applied to various PTFE com-

pounds. The model is motivated by the results of uniaxial experiments. The theoretical background is given by the theory of finite plasticity based on isomorphisms, as suggested by Bertram (1998, 2003).

The present approach is capable to describe both inelastic effects (strain rate dependence, non-linear loading and unloading, stress relaxation, retardation, creep), as described by Kletschkowski et al. (2002), and specific phenomena (SDE), as presented in Kletschkowski et al. (2003), of filled PTFE at small and finite strains. In contrast to these previous publications the endochronic viscoplastic model will be applied to characterize the temperature dependence of thermoplastic materials as well as the dependence of the stress response on the filler volume fraction. Various PTFE compounds will be taken as examples for this kind of materials.

Due to the separation of the rate-dependent and the rate-independent inelastic phenomena by an additive decomposition of the total stress response into a rate-dependent overstress and a rate-independent equilibrium stress, the presented model fits into the powerful concept of the viscoplasticity based on overstress (VBO), developed by Krempl (1996). The classical VBO, developed by Krempl, uses a constitutive decomposition of the total strain rate into elastic and inelastic parts. In order to model the elastic strain rate a linear elastic law is introduced. The inelastic strain rate is modeled in terms of the overstress which is given as the difference between the total stress and the equilibrium stress. The equilibrium stress and the additionally introduced kinematic stress are used as state variables in this concept.

Recent developments in the area of viscoplastic modeling based on overstress were published by the group of Krempl and additionally by Colak (2004). The last article is focused on the application of a small strain VBO approach to proportional and non-proportional cyclic loading. In Krempl (2001) the VBO was applied to model the relaxation properties of metallic materials. In order to incorporate non-standard rate dependence in solids, an extension of the classical VBO was published by Krempl and Ho (2002). This work is focused on rate dependent yielding, strain

softening and subsequent hardening of polymeric materials at isothermal conditions.

The findings published by Krempl and Khan (2003) give an overview of how capable the VBO concept is of modeling the rate-dependent deformation behaviour of metals and solid polymers, namely PP (polypropylene) and HDPE (high density polyethylene). In all cases the classical VBO is able to predict the rate-dependencies during loading precisely. But the classical VBO in its present form (the elastic strain rate is modeled by a linear elastic law) does not seem to be very precise in the simulation of the non-linear unloading behaviour of polymeric materials. In Krempl and Khan (2003) the classical VBO was also applied to simulate the characteristics of the short time relaxation behaviour of PP and HDPE. Unfortunately there are still noticeable differences between the experimental data and the numerical results. Furthermore only homogeneous materials were analyzed. The influence of the filler particles, introduced in order to improve the mechanical properties of polymeric materials, was not investigated.

The approach presented in this article is focused on these open questions. Therefore the non-linearities for both loading and unloading have been taken into account. The stress relaxation behaviour has been modeled for short as well as for long time relaxation processes. Finally the effect of filler particles on the mechanical properties of the investigated PTFE materials has been analyzed.

In contrast to the classical VBO, a constitutive decomposition of the total strain rate into elastic and inelastic parts is not needed in the present finite endochronic viscoplastic approach. Furthermore an endochronic flow rule is applied to describe the rate-independent phenomena. And, in contrast to previous works, the equilibrium stress contribution will be modeled by a physically non-linear elastic law. Due to these facts the proposed model could be named endochronic viscoplasticity based on overstress (eVBO).

The endochronic theory of plasticity was developed by Valanis (1971). This special theory of rate-independent material behaviour was intensively discussed by Haupt (1977, 2000) and Krawietz (1986).

2. Experiments

Various PTFE compounds have been taken as examples to study the mechanical behaviour of thermoplastic materials in isothermal uniaxial tests in both tension and compression. A typical PTFE compound (compound S, Table 1) consists of pure PTFE and a certain amount of filler particles (short cylindrical glass fibres (length ≤ 0.1 mm, diameter ≤ 0.02 mm), MoS₂). These fillers are used to improve the performing properties of pure PTFE. Glass fibres reduce wear. MoS₂ decreases friction. All compounds, which have been analyzed during the investigations, are listed in Table 1.

The experiments have been performed on an electromechanical tensile tester (TIRA 2410) in combination with a temperature chamber (NOSKE-KAESER). The tests have been focused on the measurement of loading curves at constant strain rates and the succeeding stress relaxation processes at constant strains. The dependence of the material behaviour on the temperature has been examined for the compound S. In addition this compound has been used to study the SDE. The dependence of the stress response in tension on the amount of the filler volume fraction (FVF) has been analyzed with respect to the compounds A–G.

Furthermore, stress controlled experiments (strain recovery after unloading) have been performed on compound S to verify the constitutive model by the comparison of this experimental data with the results of stress controlled numerical simulations.

Table 1
Materials used for the experiments

Compound	Glass fibre volume fraction [%]	MoS ₂ volume fraction [%]
S	5	5
A	0	0
B	5	0
C	10	0
D	15	0
E	20	0
F	25	0
G	30	0

2.1. General characteristics of PTFE compounds at ambient temperature

The explanation of the mechanical behaviour of PTFE compounds at ambient temperature starts with a few notes on previous investigations. In order to characterize the inelastic phenomena of PTFE materials, compound S has been intensively studied in uniaxial tests (tension tests at different strain rates, stress relaxation and retardation), see Pohl (1999), Klitschkowski et al. (2000) and Klitschkowski et al. (2002). Some experimental data of these investigations are shown in Fig. 1. They clarify that a successful constitutive approach to this class of materials must be able to predict the following effects:

- strain-rate-dependence during loading;
- stress relaxation at constant strain;
- partial strain recovery in the unloaded state;
- plastic deformation after unloading and strain recovery;
- asymmetric material behaviour in tension and compression (SDE).

2.1.1. Some remarks on the SDE of PTFE compounds

The behavior of pure PTFE (compound A) is nearly identical under tension and compression, but the stress–strain characteristic of PTFE compounds filled with short cylindrical glass fibres shows a considerable difference between the material behavior in tension and compression.

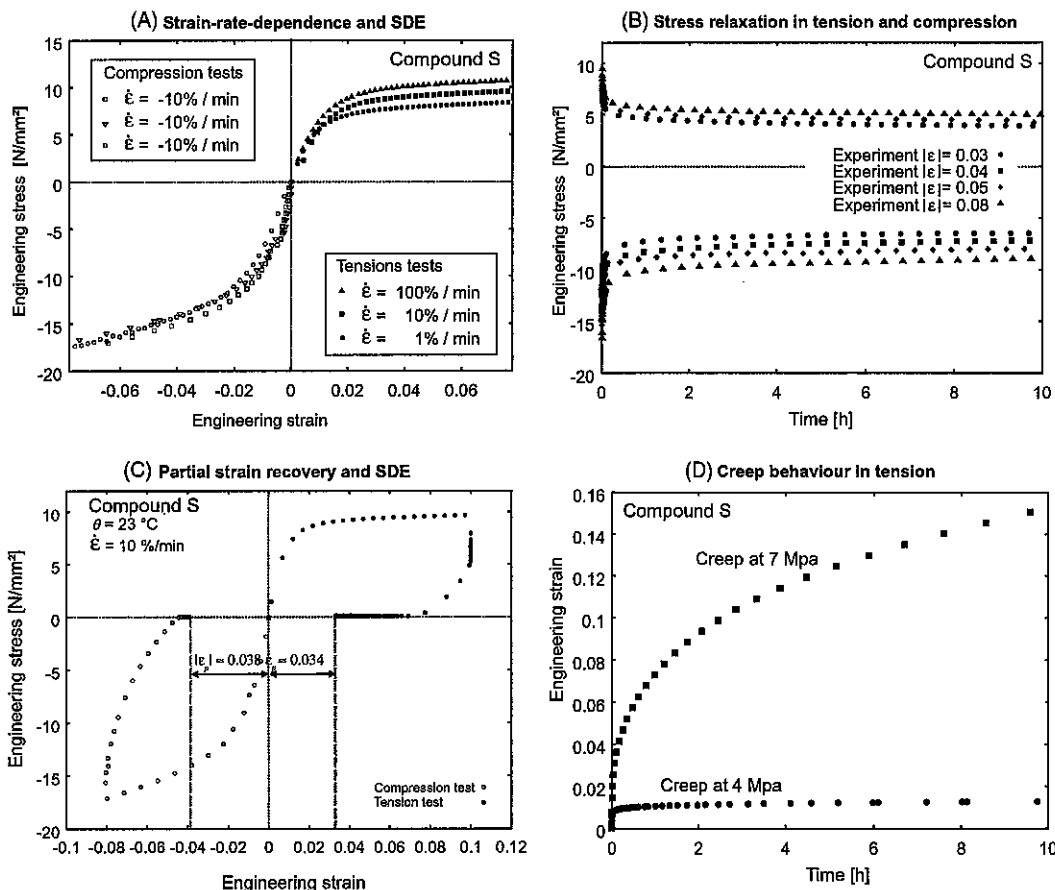


Fig. 1. Mechanical behaviour at ambient temperature.

Fig. 2(left) illustrates the SDE for isothermal loading in tension and compression at a strain rate of $1/600\text{ s}^{-1}$ by taking the compounds A, E, and S as examples. The experimental results of these tests show that

- the amount of stress in uniaxial compression is noticeably greater (up to 78%) than the amount of stress in uniaxial tension if PTFE compounds are filled with short cylindrical glass fibres;
- the amount of stress in uniaxial compression and in uniaxial tension is nearly the same if pure PTFE is analyzed;
- a higher amount of the filler volume fraction results in a softening behaviour in tension, while it has no effect on the increase of the stress response under compression, if the amount of filler material is larger than 5%.

A micromechanical explanation of the SDE is motivated by Fig. 2(right). In this figure a PTFE matrix and an embedded glass fibre is shown for simple tension and simple compression. Due to the tribological characteristics of PTFE materials, the static friction between the PTFE matrix and the glass fibre is very small. Hence simple tension results in a sliding process between the glass fibre and the matrix. Furthermore the dimension of the glass fibres (length $\leq 0.1\text{ mm}$, diameter $\leq 0.02\text{ mm}$) is small compared to the dimension of the tension test specimen ($60\text{ mm} \times 5\text{ mm} \times 2\text{ mm} = L \times T \times H$). Hence, in contrast to compos-

ites made of long fibres, there is no contribution to the stiffness of the material in tension.

The situation is completely different in simple compression. In this case, the stress is transmitted through the stiff filler material. Therefore the macroscopic stress response increases under uniaxial compression.

Furthermore, as illustrated by Fig. 1B, in both tension and compression, the stress is reduced to nearly 50% of its initial value after 10h of stress relaxation. This stress relaxation can be clearly observed under tension as well as under compression. The engineering approach suggested by Pohl (1999) neglects the stress relaxation in compressed parts of the seal. Hence it is not adequate to model the material behavior of filled PTFE under compression. In order to overcome this problem, viscoplastic material models of the over-stress type are proposed in Kletschkowski et al. (2001, 2003). These models allow to simulate stress relaxation processes under tension as well as under compression. In the present work, the approach, presented in Kletschkowski et al. (2003), will be applied to model the SDE of PTFE compounds at finite strains.

2.2. Dependence of the material behaviour on the temperature

As illustrated by the experimental data shown in Fig. 4A1–A3, the mechanical behaviour of PTFE compounds depends on temperature. In order to analyze this dependence, uniaxial tension

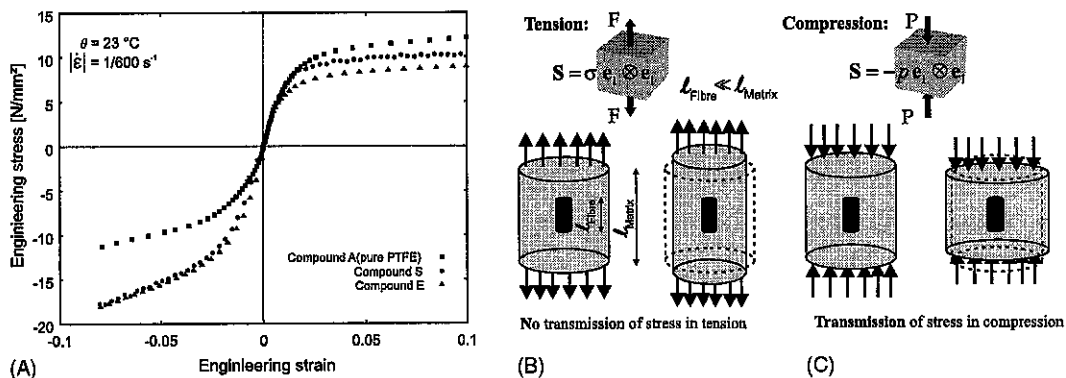


Fig. 2. Mechanical behaviour under tension and compression.

tests at various constant temperatures (23°C, 50°C, 80°C, 100°C and 120°C) have been performed. Compound S has been used for this test series. In a first load step the test specimens have been strained at a constant strain rate of $\dot{\epsilon} = 1/60\text{s}^{-1}$. Then, in a second load step, the strain has been kept constant at $\epsilon_{\text{max}} = 0.08$.

The results, shown in Fig. 4A1, show that an increase in temperature results in a decrease of the measured stress response during loading. While the initial slope of the loading curves decreases with increasing temperature, the differences due to different loading curves are nearly constant in the yield range. Furthermore, as documented by Fig. 4A2 and A3, a higher constant temperature results in a lower stress level during the stress relaxation process. The differences between different relaxation curves become smaller at higher temperatures.

2.3. Dependence of the material behaviour on filler volume fraction

As reported in a previous section, an increase of the filler volume fraction results in a softening of PTFE compounds in simple tension. In order to study the dependence of the material behaviour on the amount of glass fibre particles, uniaxial tension tests have been performed on the compounds A–G.

These tests have been performed at ambient temperature. In a first load step the test specimens have been strained at a constant strain rate of $\dot{\epsilon} = 1/600\text{s}^{-1}$. In a second load step the strain has been kept constant at $\epsilon_{\text{max}} = 0.1$.

The results of these experiments are shown in Fig. 7. They show that an increase of the glass fibre volume fraction results in a softening behaviour under simple tension. While the filler volume fraction does not effect the initial slope of the loading curve, the stress response in the yield range depends clearly on the amount of filler particles. In this range the loading curves of the compounds B–G are nearly parallel.

Moreover, as documented by Fig. 7(right), a higher amount of the filler volume fraction results in a lower stress level during the stress relaxation process under simple tension.

3. Constitutive modeling

The endochronic viscoplastic material model, presented by Kletschkowski et al. (2003), is capable to describe the inelastic phenomena of PTFE compounds observed in the experiments. Therefore the fundamental ideas of this approach will be summarized. Afterwards a finite rate-dependent endochronic approach (including a concept to model the SDE) will be presented. This constitutive model will be applied to simulate the dependence of the model parameters on the temperature and the filler volume fraction.

3.1. Motivation by a rheological model

The finite rate-dependent endochronic approach, which we propose, can be easily motivated by a simple rheological model of overstress type.

This model consists of a rate-independent endochronic model in parallel connection with a non-linear Maxwell model, see Fig. 3. The endochronic model is used to describe rate-independent material behaviour with equilibrium hysteresis. The rate-dependent Maxwell model is needed to describe time-dependent effects.

The rheological model motivates an additive decomposition of the total stress response σ into an equilibrium stress σ_{∞} and an overstress σ_{ov} . Furthermore, internal variables are needed to describe rate-dependent (ϵ_{iv}) and rate-independent (ϵ_{ip}) inelastic phenomena.

In order to model non-linear behaviour in both loading and unloading, a logarithmic elastic law is applied for the equilibrium stress.

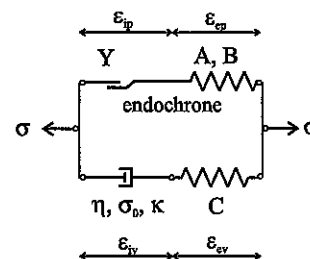


Fig. 3. Rheological model of the overstress type.

$$\sigma_\infty = A \ln(1 + B\langle \varepsilon_{ep} \rangle) - A \ln(1 - B\langle -\varepsilon_{ep} \rangle)$$

with

$$\langle x \rangle := \frac{x + |x|}{2} \tag{1}$$

A and B are the elastic constants of the equilibrium branch. At very small strains ($\varepsilon \leq 0.01$) Eq. (1) can be replaced by Hooke's law. A linear elastic law with the model parameter C is used to model the overstress.

$$\sigma_{ov} = C\varepsilon_{ev} \tag{2}$$

In addition to the elastic laws, ordinary differential equations are introduced to describe the evolution of the internal variables. A rate-independent endochronic flow rule is applied to model the evolution of the internal variable ε_{ip} .

$$\dot{\varepsilon}_{ip} = \frac{1}{Y} \sigma_\infty |\dot{\varepsilon}| \tag{3}$$

Applications of this kind of evolution equations on non-linear material modeling can be found in Krawietz (1986). As explained by Kletschkowski et al. (2002), the material parameter Y represents the maximum stress that can be reached in the endochronic element during a deformation process. The evolution of the internal variable ε_{iv} is described by a highly non-linear differential equation of the Garofalo-type.

$$\dot{\varepsilon}_{iv} = \frac{1}{\eta} \left[\sinh \left(\frac{\sigma_{ov}}{\sigma_0} \right) \right]^\kappa \tag{4}$$

η , κ and σ_0 are model parameters. In combination with an additive decomposition of the total strain ε into elastic and inelastic parts

$$\varepsilon = \varepsilon_{ep} + \varepsilon_{ip} = \varepsilon_{ev} + \varepsilon_{iv} \tag{5}$$

the rheological model is capable to describe the viscoplastic mechanical behaviour of thermoplastic materials at moderate strains and isothermal conditions. Only six constant model parameters are needed. The main ideas of this approach are

- the additive decomposition of the total stress response;
- the application of a non-linear elastic law to model the equilibrium stress;
- the description of the rate-dependent effects by an endochronic flow rule.

The application of this rheological model to filled PTFE is documented by Kletschkowski et al. (2002).

3.2. Finite endochronic viscoplasticity based on overstress

3.2.1. Theoretical framework

The theoretical background of the endochronic viscoplastic material model for filled PTFE, proposed by Kletschkowski et al. (2002), is given by a modification of Bertram's approach to finite plasticity based on material isomorphisms, see Bertram (1998, 2003). Therefore the fundamental ideas of this concept have to be summarized.

This approach starts with the definition of the independent and the dependent variables of a deformation process. If \mathbf{F} is the deformation gradient and \mathbf{T} the Cauchy stress tensor, then $\mathbf{C} = \mathbf{F}^T \mathbf{F}$ defines the right Cauchy–Green tensor and $\mathbf{S} = \mathbf{F}^{-1} \mathbf{T} \mathbf{F}^{-T}$ the material stress tensor. \mathbf{S} is work-conjugated to \mathbf{C}

$$\frac{1}{\rho} \text{tr}(\mathbf{T}\mathbf{L}) = \frac{1}{2\rho} \text{tr}(\mathbf{S}\dot{\mathbf{C}}) \tag{6}$$

$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ denotes the spatial velocity gradient. An elastic range $\{E_p, h_p\}$ is defined as a path-connected subset E_p of the configuration space which forms a differentiable manifold with a boundary (Bertram, 1998). h_p is the elastic law in the elastic range E_p . This approach to finite plasticity is completed by the assumptions that

- the material state is in an elastic range at any time;
- the elastic range is only changed during an inelastic deformation process;
- the elastic laws of all elastic ranges of the material are isomorphic.

The change of the initial elastic range $\{E_0, h_0\}$ to the actual elastic range $\{E_p, h_p\}$ is described by an elastic isomorphism \mathbf{P} according to

$$h_p(\mathbf{C}) = \mathbf{P}h_0(\mathbf{P}^T \mathbf{C} \mathbf{P})\mathbf{P}^T =: \mathbf{P}h_0(\mathbf{C}_{ep})\mathbf{P}^T \tag{7}$$

\mathbf{P} is called plastic transformation. Its properties are discussed in detail in Bertram (1998, 2003).

The evolution of \mathbf{P} is described by an ordinary differential equation in the context of finite plasticity.

3.3. A rate-dependent endochronic constitutive model at finite strains

The endochronic viscoplastic material model for filled PTFE (Klitschkowski et al., 2002) is based on Bertram's approach to finite plasticity. In the present work, it will be combined with an extension of the endochronic viscoplastic model for materials with different behaviour in tension and compression (Klitschkowski et al., 2003) in order to describe the SDE of PTFE compounds at finite strains. The present approach is based on the following assumptions:

- the material has isomorphic elastic ranges, and the inelastic deformation has no effect on the elastic properties;
- the evolution of an inelastic transformation \mathbf{P} is determined by associated flow rules using a J_2 -theory of inelastic flow;
- due to the statistical distribution of the short cylindrical glass fibres all materials will be assumed to be isotropic on the macroscopic scale;
- the volumetric changes are assumed to be purely elastic.

In order to model a purely elastic stress response due to the volumetric parts of the deformation, a multiplicative decomposition of the right Cauchy–Green tensor \mathbf{C} is performed.

$$\mathbf{C} = (\det \mathbf{C})^{\frac{1}{3}} \bar{\mathbf{C}}, \mathbf{C}_{\text{vol}} := (\det \mathbf{C})^{\frac{1}{3}} \mathbf{I}, \bar{\mathbf{C}} := (\det \mathbf{C})^{-\frac{1}{3}} \mathbf{C} \quad (8)$$

The idea of an additive decomposition of the material stress response, explained by the rheological model, is transferred to three dimensions. Hence the material stress tensor \mathbf{S} is splitted into a purely elastic stress \mathbf{S}_{vol} (due to changes in the volume) a deviatoric equilibrium stress $\mathbf{S}_{\infty}^{\text{DEV}}$ and a deviatoric overstress $\mathbf{S}_{\text{ov}}^{\text{DEV}}$.

$$\begin{aligned} \mathbf{S} &= \mathbf{S}_{\text{vol}} \mathbf{C}^{-1} + \mathbf{S}_{\infty}^{\text{DEV}} + \mathbf{S}_{\text{ov}}^{\text{DEV}} \iff \mathbf{T} = \mathbf{F} \mathbf{S} \mathbf{F}^{\text{T}} \\ &= \mathbf{T}_{\text{vol}} \mathbf{I} + \mathbf{T}_{\infty}^{\text{dev}} + \mathbf{T}_{\text{ov}}^{\text{dev}} \end{aligned} \quad (9)$$

While $(\cdot)^{\text{DEV}}$ denotes the material deviatoric operation, as defined by Eq. (10), $(\cdot)^{\text{dev}}$ denotes the classical deviatoric operation given by Eq. (11).

$$\mathbf{S}^{\text{DEV}} = \mathbb{N}[\mathbf{S}] \quad \text{with } \mathbb{N} := \left(\mathbb{I} - \frac{1}{3} \mathbf{C}^{-1} \otimes \mathbf{C} \right) \quad (10)$$

$$\mathbf{T}^{\text{dev}} = \mathbb{D}[\mathbf{T}] \quad \text{with } \mathbb{D} := \left(\mathbb{I} - \frac{1}{3} \mathbf{I} \otimes \mathbf{I} \right) \quad (11)$$

The following relation holds between these different deviatoric operations:

$$\mathbf{S}^{\text{DEV}} \mathbf{C} = (\mathbf{S} \mathbf{C})^{\text{dev}} \quad (12)$$

The volumetric stress response is given by a non-linear elastic law Eq. (13), as proposed by Haupt (2000) in an Eulerian description of finite elasticity. The model parameter K can be interpreted as the bulk modulus of the material.

$$S_{\text{vol}} = K (\det \mathbf{C})^{-\frac{1}{3}} \ln \{ \sqrt{\det \mathbf{C}} \} \quad (13)$$

An elastic model of Neo–Hooke type is applied to describe the overstress. The model parameter G_{ov} can be interpreted as the shear modulus of the overstress branch.

$$\begin{aligned} \mathbf{S}_{\text{ov}}^{\text{DEV}} &= 2G_{\text{ov}} (\det \mathbf{C})^{-\frac{2}{3}} \mathbb{N}[\mathbf{P}_{\text{V}} (\bar{\mathbf{C}}_{\text{ev}} - \mathbf{I}) \mathbf{P}_{\text{V}}^{\text{T}}], \\ \bar{\mathbf{C}}_{\text{ev}} &= \mathbf{P}_{\text{V}}^{\text{T}} \bar{\mathbf{C}} \mathbf{P}_{\text{V}} \end{aligned} \quad (14)$$

\mathbf{P}_{V} is called viscous transformation. This internal variable is used to model rate-dependent inelastic phenomena of thermoplastic materials.

The generalization of the logarithmic elastic law for the equilibrium stress to three dimensions and finite strains is given by Eqs. (15)–(17). The non-linear relation $h_{\infty}(\cdot)$ is applied to the eigenvalues of $\bar{\mathbf{C}}_{\text{ep}}$. The model parameter G_{∞} can be interpreted as the shear modulus of the equilibrium branch. B is another elastic model parameter, which is needed for this non-linear stress–strain relation.

$$\begin{aligned} \mathbf{S}_{\infty}^{\text{DEV}} &= 2G_{\infty} (\det \mathbf{C})^{-\frac{2}{3}} \mathbb{N} \left[\mathbf{P}_{\text{P}} \left(\sum_{i=1}^3 s_i \mathbf{n}_i \otimes \mathbf{n}_i \right) \mathbf{P}_{\text{P}}^{\text{T}} \right], \\ \bar{\mathbf{C}}_{\text{ep}} &= \mathbf{P}_{\text{P}}^{\text{T}} \bar{\mathbf{C}} \mathbf{P}_{\text{P}} \end{aligned} \quad (15)$$

$$s_i = \ln(1 + B \langle \lambda_i \rangle) - (1 + D) \ln(1 - B \langle -\lambda_i \rangle) =: h_{\infty}(\lambda_i) \quad (16)$$

$$(\lambda_i, \mathbf{n}_i) = \text{eigensystem } [\bar{\mathbf{C}}_{\text{ep}} - \mathbf{I}] \quad (17)$$

\mathbf{P}_P is called plastic transformation. This internal variable is used to model rate-independent inelastic phenomena of thermoplastic materials. In contrast to the constant model parameters G_∞ and B , the scalar internal variable D is introduced to model the SDE of fibre filled materials.

The evolution of D depends on the state of stress. In order to distinguish between tension and compression, an indicator function ξ , based on the invariants of $\mathbf{S}_\infty^{\text{DEV}} \mathbf{C}$, is defined as

$$\xi := \frac{\sqrt{27}}{2} \frac{J_3}{(J_2)^{3/2}}, \quad J_i := \frac{1}{i} \text{tr}[(\mathbf{S}_\infty^{\text{DEV}} \mathbf{C})^i] \quad (18)$$

The indicator function ξ enables the introduction of stress mode factors for tension (w_t) and compression (w_c).

$$w_t = \frac{1}{2}(1 + \xi), \quad w_c = \frac{1}{2}(1 - \xi) \quad (19)$$

The evolution of D is assumed to be driven by the equivalent material equilibrium stress σ_{VP} , as defined by

$$\sigma_{VP} = \sigma_{VP}(\mathbf{S}_\infty^{\text{DEV}} \mathbf{C}) := \sqrt{\frac{3}{2} \text{tr}\{(\mathbf{S}_\infty^{\text{DEV}} \mathbf{C})(\mathbf{S}_\infty^{\text{DEV}} \mathbf{C})\}} \quad (20)$$

Due to the permanent and the non-permanent characteristics of the SDE of filled PTFE, as reported by Kletschkowski et al. (2001, 2003), an additive decomposition of \dot{D} is performed

$$\dot{D} = \dot{D}_p + \dot{D}_n \quad (21)$$

Non-linear ordinary differential equations of the saturation type are applied to model the evolution of the permanent and the non-permanent parts of D .

$$\dot{D}_p = K_1 w_c \sigma_{VP} \dot{\epsilon}_V (D_p^{\text{max}} - D_p) \quad (22)$$

$$\dot{D}_n = K_2 w_c \sigma_{VP} \dot{\epsilon}_V (D_n^{\text{max}} - D_n) - w_c K_3 (D_n)^{K_4} \quad (23)$$

K_i ($i = 1, \dots, 4$), D_p^{max} , and D_n^{max} are non-negative model parameters. Eqs. (22) and (23) show that the evolution of D is possible only under compression.

The development of D_p is limited by D_p^{max} . The evolution of D_n stops if D_n^{max} is reached. In contrast to Eq. (22), a static recovery process is allowed by Eq. (23).

Finally the evolution equations for the viscous and plastic internal variables have to be generalized to three dimensions. Hence a highly non-linear flow rule of the Garofalo-type with three constant model parameters (η, κ, σ_0) is applied Eq. (24) for the evolution of the viscous transformation \mathbf{P}_V

$$\dot{\mathbf{P}}_V \mathbf{P}_V^{-1} = -\frac{3}{2\eta} \left[\sinh \left(\frac{\sigma_{VV}}{\sigma_0} \right) \right]^\kappa \frac{\mathbf{S}_{ov}^{\text{DEV}} \mathbf{C}}{\sigma_{VV}} \quad (24)$$

The direction of the viscous flow is given by the deviatoric part of the overstress $\mathbf{S}_{ov}^{\text{DEV}} \mathbf{C}$. Its amplitude is defined by a scalar valued constitutive equation based on the equivalent material overstress σ_{VV} (Eq. (25)).

$$\sigma_{VV} = \sigma_{VV}(\mathbf{S}_{ov}^{\text{DEV}} \mathbf{C}) := \sqrt{\frac{3}{2} \text{tr}\{(\mathbf{S}_{ov}^{\text{DEV}} \mathbf{C})(\mathbf{S}_{ov}^{\text{DEV}} \mathbf{C})\}} \quad (25)$$

The evolution of the plastic transformation \mathbf{P}_P is determined by a rate-independent endochronic flow rule (Eq. (13)).

$$\dot{\mathbf{P}}_P \mathbf{P}_P^{-1} = -\frac{3}{2} \left(\frac{w_t}{Y} + \frac{w_c}{Y(1+D)} \right) \sigma_{VP} \dot{\epsilon}_V \frac{(\mathbf{S}_\infty^{\text{DEV}} \mathbf{C})}{\sigma_{VP}} \quad (26)$$

Y is a stress type model parameter. The direction of the plastic flow is given by the deviatoric part of the equilibrium stress $\mathbf{S}_\infty^{\text{DEV}} \mathbf{C}$. Its amplitude is determined by a scalar valued constitutive equation, based on the equivalent material equilibrium stress σ_{VP} .

The SDE comes into picture with the introduction of the internal variable D into the elastic law for the equilibrium stress Eq. (16). But the macroscopic plastic deformation in the PTFE matrix differs not significantly if tension or compression is applied. This conclusion can be drawn, if the hysteresis loop for simple tension is compared to the hysteresis loop under simple compression (Fig. 1C). The small deviation can be explained by the fact that the retardation process under simple

compression was measured for 60 s. The retardation process in simple tension was observed up to end of the viscous strain recovery process. Furthermore this small difference between the plastic strain in tension and compression is within the range of the experimental scatter. Therefore, in order to avoid an overestimation of plastic strains (this problem occurs if only the stress increase under simple compression has been taken into account in order to model the SDE by a pressure dependent plastic flow rule, see Pohl (1999)), the indicator functions w_t and w_c and the internal variable D are used to get rid of the SDE in the plastic strain rate. Hence Eq. (26) shows that the evolution of the plastic transformation is not affected by the SDE.

The presented endochronic approach to finite viscoplasticity based on overstress Eqs. (15)–(26) is capable to describe finite inelastic deformations of isotropic materials without any constitutive decomposition of the total deformation into elastic and inelastic parts. Due to the Lagrangian approach, all time derivatives are material time derivatives. Hence all evolution equations and the elastic laws are objective. In general the inelastic transformations \mathbf{P}_P and \mathbf{P}_V are non-symmetric tensors. Hence the inelastic spin is included into the concept. The Cauchy stress at any time can be easily calculated by the simple push forward operation $\mathbf{T} = \mathbf{FSF}^T$.

4. Identification and numerical results

The finite rate-dependent endochronic constitutive model, presented in the previous section, has been identified using experimental data of strain controlled uniaxial tests (loading and succeeding stress relaxation). Next, the identification procedure will be described in general. Then the application of the model due to special points of interest will be shown.

The identification process consists of three steps. These are the

- preparation of the numerical identification;
- identification of the constitutive model;

- quality control (comparison of numerical results and experimental data).

The numerical identification process has been prepared by fitting the measured stress response to analytical functions. Loading curves have been adjusted to

$$\sigma_c(t) = l_1 t + l_2 \tanh(l_3 t) \quad (27)$$

and the stress relaxation curves to

$$\sigma_c(t) = r_1(t^2) + r_3(t^4) \quad (28)$$

Due to the fact that the measured quantities are engineering strain ε and engineering stress σ , the uniaxial Cauchy stress has been calculated by

$$\sigma_c = \sigma(1 + \varepsilon). \quad (29)$$

If the total deformations are moderate, Eq. (29) gives a sufficient approximation also for low-compressible thermoplastic materials like PTFE.

In order to identify the model parameters by a numerical identification procedure, the Levenberg–Marquardt algorithm has been applied to minimize

$$\chi^2 = \sum_{i=1}^N [\sigma_c(t_i) - T_{11}(t_i)]^2 \rightarrow \text{MIN} \quad (30)$$

the differences between the measured and calculated values of the longitudinal Cauchy stress. All ordinary differential equations have been solved by an explicit predictor-corrector procedure of order 2 with constant time stepping. Finally the quality of model prediction has been checked.

Each numerical identification of the model parameters has been performed at a moderate state of strain ($\varepsilon_{\max} \leq 0.1$). Therefore the definitions listed below have been used to reduce the number of unknown model parameters.

$$\begin{aligned} A &:= 2G_{\infty}(1 + \nu), & C &:= 2G_{\text{ov}}(1 + \nu), \\ K &:= \frac{AB + C}{3(1 - 2\nu)} \end{aligned} \quad (31)$$

Due to the results of the experimental investigations on PTFE materials, presented by Pohl (1999), Poisson's ratio ν has been set to 0.46 in all cases.

4.1. Dependence of the model parameters on the temperature under simple tension

In order to identify the dependence of the constitutive model on the temperature, the model parameters (A, C, Y and σ_0) have been adjusted for the compound S using uniaxial tension tests at various constant temperatures. All tests have been performed at a strain rate of $\dot{\varepsilon} = 1/600 \text{ s}^{-1}$. Every succeeding stress relaxation has been measured at $\varepsilon_{\max} = 0.08$.

In contrast to these four parameters, the constants B, η and κ have been identified only once at 23°C. Afterwards these three parameters have been kept constant at all higher temperatures. Hence each component of the constitutive model contains one temperature dependent parameter. While A describes the temperature dependence of the elastic properties of the equilibrium branch, C contains the temperature dependence of the overstress branch. Y characterizes the dependence of the plastic flow on temperature, and σ_0 is used to model the temperature dependence of the viscous flow.

The results of this identification process are listed in Table 2. Fig. 4A1–A3 demonstrates that the numerical results obtained by using the identified parameters, are in good agreement with the experimental data in both short and long time periods.

Fig. 5 illustrates the dependence of the model parameters on the temperature. In order to illustrate these dependencies, the values of A, C, Y and σ_0 , identified at 23°C, 50°C, 80°C, 100°C and 120°C, have been adjusted to analytical functions, shown in Fig. 5. Each function is decreasing under an increase in temperature.

4.2. Verification using simulations of stress controlled unloading

In order to demonstrate the capability of the presented endochronic viscoplastic approach to describe the material non-linearities during stress controlled unloading Fig. 6a illustrates the stress–strain-characteristic of compound S (triangles) and of the rheological model (solid line). The model parameters were identified as $A = 1.6 \text{ N/mm}^2$, $Y = 10.5 \text{ N/mm}^2$, $\eta = 9.0 \times 10^5 \text{ 1/s}$, $\kappa = 2.4$, $B = 440.0$, $C = 304.0 \text{ N/mm}^2$, $\sigma_0 = 1.3 \text{ N/mm}^2$ as reported in Kletschkowski et al. (2002).

Because of the non-linear elastic law for the equilibrium stress (Eq. (1)) and due to the inelastic flow rules (Eqs. (3) and (4)) the presented approach is capable of describing both the mechanical behaviour of PTFE compounds during strain controlled loading and the mechanical behaviour of these materials during stress controlled unloading using only one set of model parameters.

The numerical identification of the finite endochronic viscoplastic model at 23°C has been verified by a simulation of a stress controlled deformation process (strain recovery in the unloaded state) without any readjustment of model parameters. Therefore, in the first (strain controlled) part of the simulation, simple tension at $\dot{\varepsilon} = 1/600 \text{ s}^{-1}$ and the succeeding stress relaxation at $\varepsilon = 0.1$ have been evaluated. In the second (stress controlled) part of the simulation the material has been unloaded at $\dot{\sigma} = -0.5 \text{ (N/mm}^2\text{)s}^{-1}$. Afterwards the strain recovery process (retardation) has been computed at $\sigma = 0$ for 10h. Engineering strain has been calculated by the use of the following relation:

Table 2
Dependence of the model parameters on temperature

θ [°C]	23	50	80	100	120
A [N/mm ²]	1.48	1.2	1.03	0.95	0.92
B	127.23	127.23	127.23	127.23	127.23
Y [N/mm ²]	9.75	8.30	7.35	6.99	6.63
C [N/mm ²]	252.41	210.55	111.74	54.16	29.97
η [s ⁻¹]	7.65054×10^5	7.65054×10^5	7.65054×10^5	7.65054×10^5	7.65054×10^5
κ	2.2	2.2	2.2	2.2	2.2
σ_0 [N/mm ²]	1.3	0.84	0.52	0.36	0.24

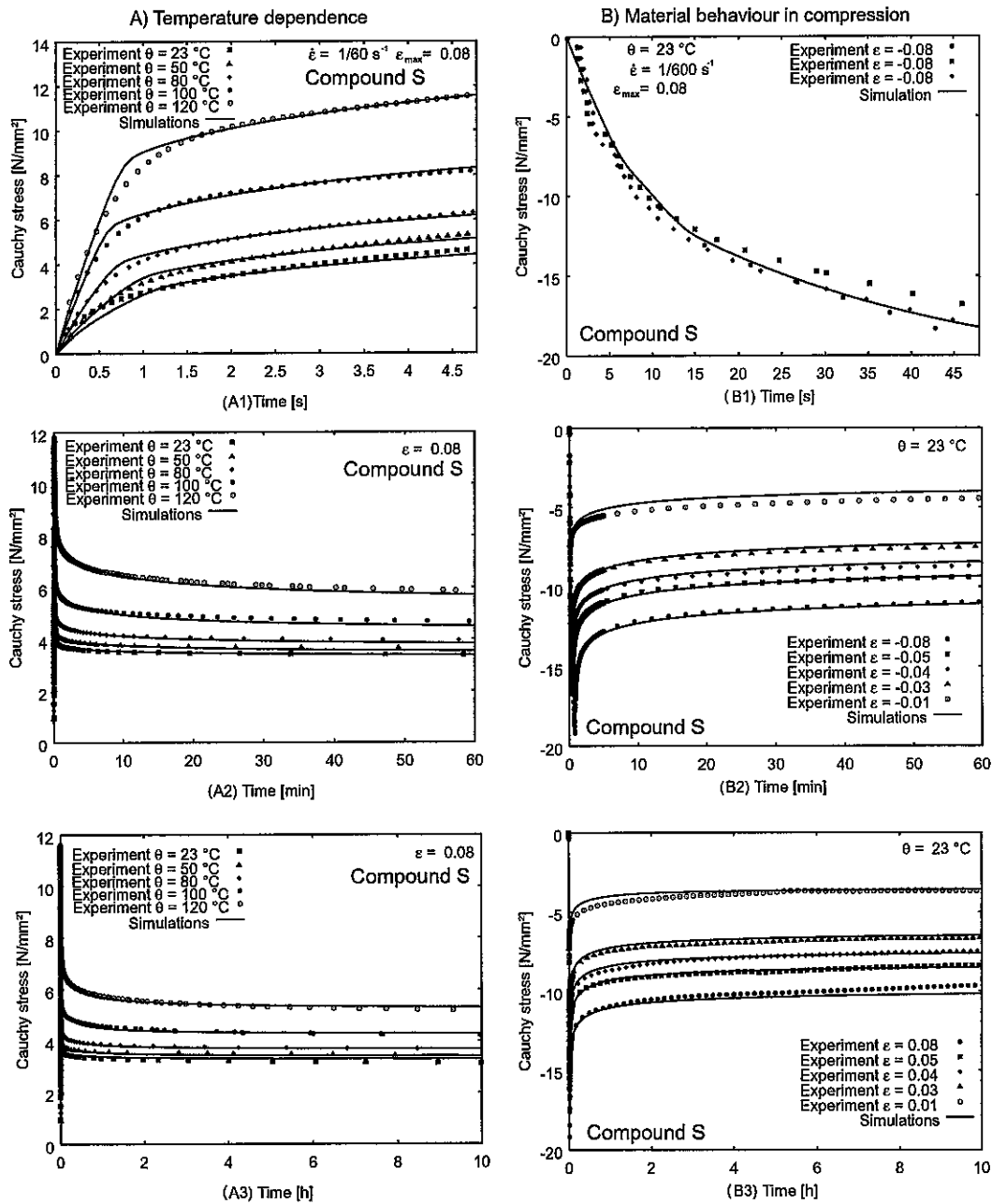


Fig. 4. Temperature dependence and mechanical behaviour under compression.

$$\epsilon = F_{II} - 1 \tag{32}$$

Fig. 6b and c illustrates that the numerical results again are in fair agreement with the experimental data. The finite endochronic viscoplastic model is

very precise in the simulation of the reduction of the total engineering strain ϵ during stress controlled unloading as well as in the calculation of the short and the long time retardation behaviour of the investigated PTFE compound in unloaded

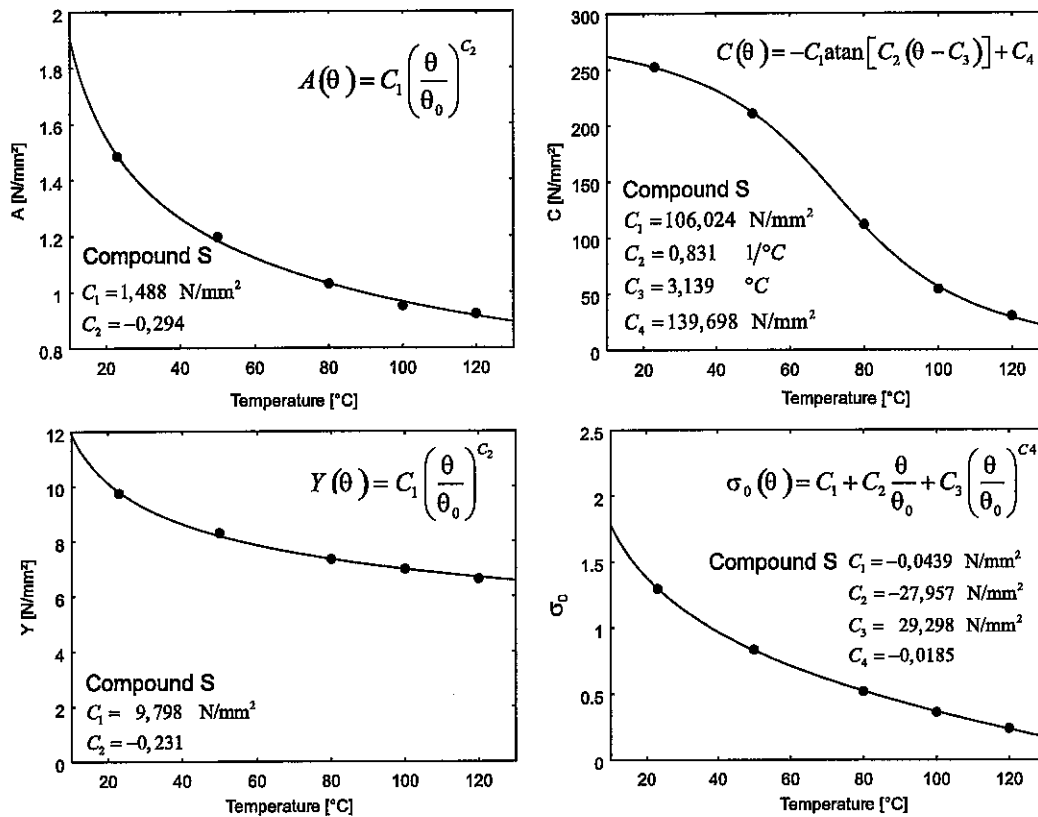


Fig. 5. Dependence of the model parameters on temperature.

state. Hence the proposed model is capable of describing the non-linearities during stress controlled unloading.

4.3. Identification of the SDE parameters under simple compression

In order to identify the dependence of the constitutive model under simple compression, a uniaxial compression test at a strain rate of $\dot{\epsilon} = 1/600 \text{ s}^{-1}$ and the succeeding stress relaxation at $\epsilon_{\max} = -0.08$ have been used.

All model parameters, which have been identified in tension, have been kept constant. Hence the differences between the measured data and the numerical results have been minimized by a variation of K_i ($i = 1, \dots, 4$), D_p^{\max} and D_n^{\max} . The results are listed in Eqs. (33) and (34).

$$K_1 = 2.47, \quad K_2 = 18.1, \quad K_3 = 1.4 \times 10^{-7}, \quad K_4 = 14.97 \quad (33)$$

$$D_p^{\max} = 1.29, \quad D_n^{\max} = 2.90 \quad (34)$$

In order to check the capability of the presented model of describing the SDE of PTFE compounds the experimental data of the compression tests shown in Fig. 1A and B have been compared to results of numerical simulations of simple compression tests.

Fig. 4B1–B3 presents the experimental data (originally shown in Fig. 1A and B) together with the numerical results. In Fig. 4B1 the engineering stress is plotted versus engineering strain for compression loading. Fig. 4B2 and B3 illustrates the measured and calculated curves of the succeeding stress relaxation under compression after loading at a strain rate of $\dot{\epsilon} = 10\%/min$. In Fig. 4B2 the

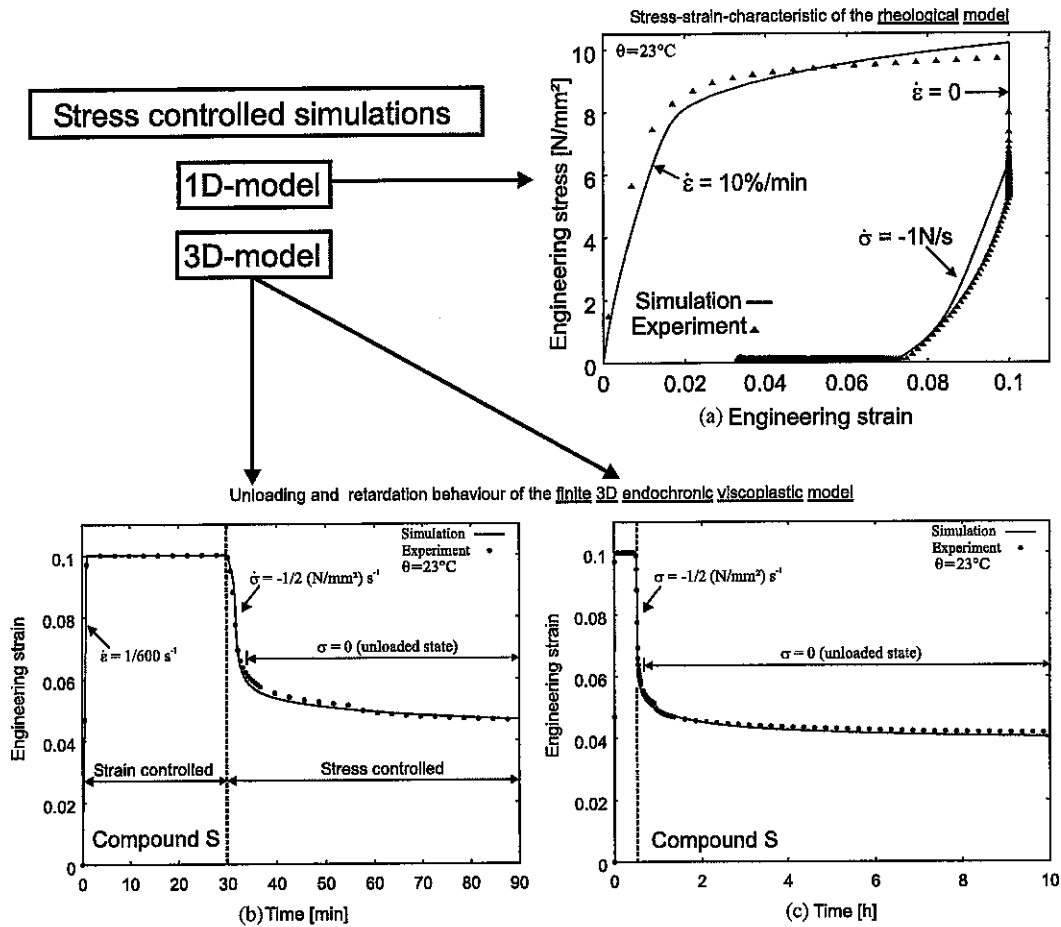


Fig. 6. Strain recovery: Experimental data and numerical results.

short time relaxation behaviour of the investigated PTFE compound is compared to the numerical results. The long term relaxation behaviour is shown in Fig. 4B3.

Fig. 4B1–B3 illustrates that the proposed model is capable of predicting the material response of the analyzed PTFE compound at various strain levels in both short and long time processes. The agreement between experimental data and numerical results is satisfactory. The model is very precise in the calculation of the long time material behavior during the stress relaxation processes, see Fig. 4B3. The numerical results are in sufficient agreement with the short time relaxation data, too, as illustrated by Fig. 4B2. Hence the proposed model is capable of describing both the instantane-

ous viscous flow after loading and the long term relaxation behaviour of the analyzed material.

The small deviations in tracking the compression response, see Fig. 4B1, are due to measuring errors caused by the more or less statistical distribution of the filler materials. Nevertheless the results of the numerical computations are within the range of the experimental scatter.

As illustrated by Fig. 2C, an increase in the glass fibre volume fraction produces no further increase in the stress response under simple compression. The experimental data of the compression tests, performed on compound S and on compound E, lie closely together. Hence it is possible to conclude that a successful simulation of the SDE for compound S (shown in Fig. 4B1) allows

a satisfactory prediction of the SDE that has been observed for compound E using Eqs. (33) and (34) as the set for the SDE model parameters.

Due to this fact it is sufficient to study the dependence of the model parameters on the filler volume fraction in tension. The results of these investigations will be presented in the following section.

4.4. Model parameters and filler volume fraction in simple tension

The experimental data of tension tests performed on the PTFE compounds A–G at a strain rate of $\dot{\epsilon} = 1/600\text{s}^{-1}$ and the measured curves of the succeeding stress relaxation processes at $\epsilon_{\text{max}} = 0.1$, measured for 120 min, have been used to identify the dependence of the model parameters A, B, Y, C, η, κ and σ_0 on the filler volume fraction in simple tension. Experiments and simulations have been restricted to ambient temperatures.

The results of this identification process are listed in Table 3. Fig. 7 demonstrates that the numerical results, obtained by applying the parameters listed in Table 3, are in good agreement with the experimental data under both loading and stress relaxation.

In order to quantify the dependence of A, B, Y, C, η, κ and σ_0 on the filler volume fraction, the values of these parameters, identified for different compounds, have been fitted to linear equations. Fig. 8 illustrates the results of these adjustments. They can be summarized in the following statements.

- The filler volume fraction affects the elastic constants of the constitutive model as well as the inelastic parameters.
- While the variation of A, C and η is nearly linear (Fig. 8A, C, D), the dependence of the model parameters σ_0 and κ on the filler volume fraction is non-linear.
- The model parameter B varies around a constant average value.

Table 3
Dependence of the parameters on the fibre volume fraction (FVF)

FVF[%]	0	5	10	15	20	25	30
A [N/mm ²]	1.51	1.07	1.05	0.77	0.93	0.74	0.74
B	91.26	127.99	105.95	125.1	91.77	108.09	113.38
Y [N/mm ²]	9.75	9.75	9.75	9.75	9.75	9.75	9.75
C [N/mm ²]	164.24	172.84	163.29	178.02	178.70	200.20	184.68
η [10 ⁵ s ⁻¹]	2.44	3.37	6.02	11.16	10.12	13.5	12.3
κ	2.59	2.67	1.83	1.41	1.47	1.47	1.55
σ_0 [N/mm ²]	1.91	1.72	1.06	0.75	0.76	0.7	0.67

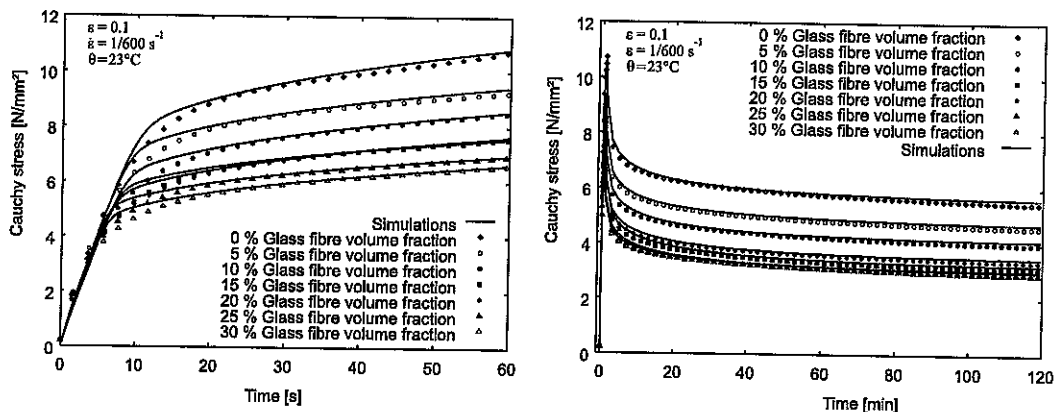


Fig. 7. Dependence of the material behaviour on the filler volume fraction.

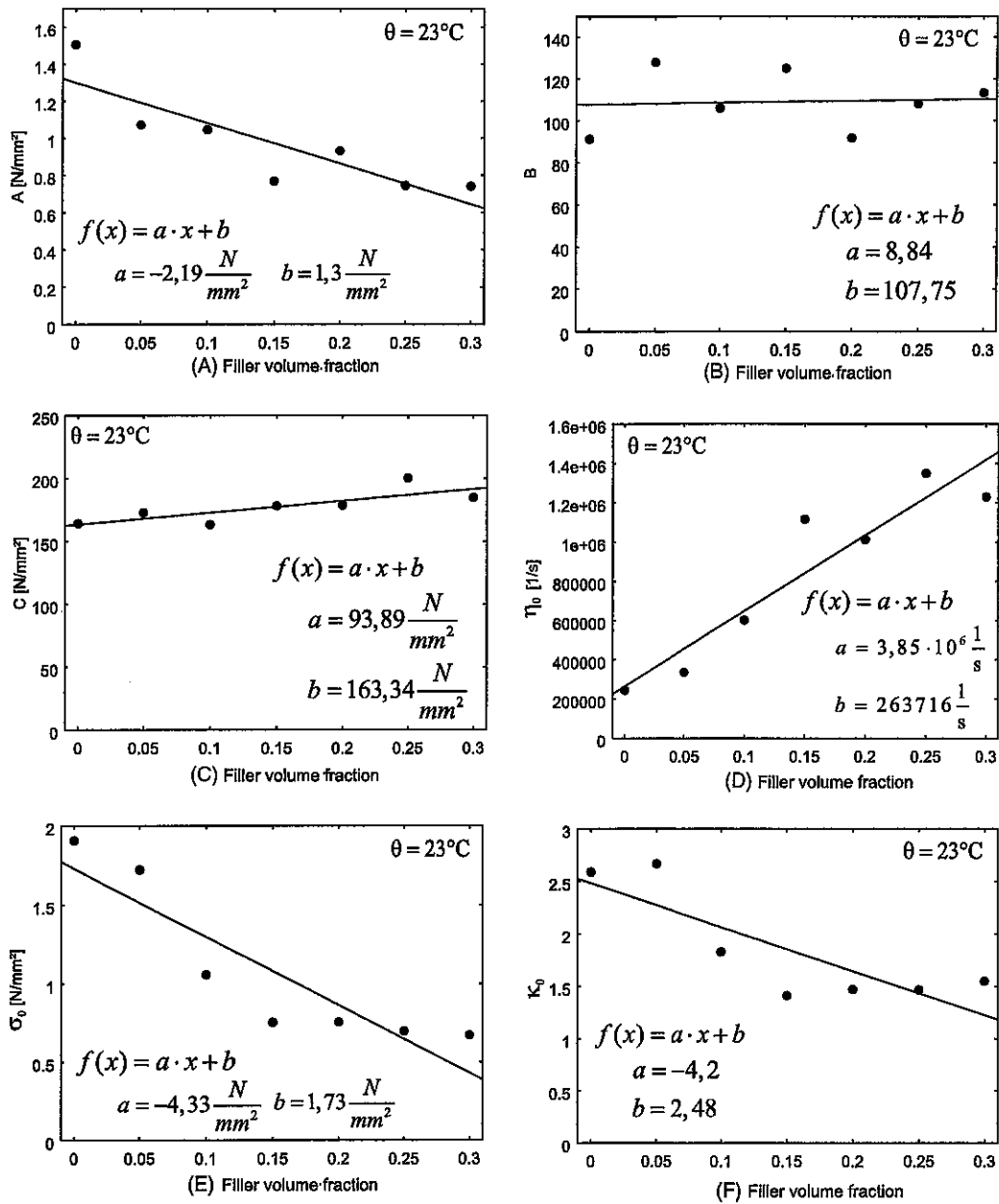


Fig. 8. Dependence of the model parameters on the filler volume fraction.

4.5. Some comments on the sensitivity of the model parameters

In order to identify the material constants the Levenberg–Marquardt algorithm has been ap-

plied. This method has the advantage of being systematic. On the other hand it is reasonable to comment on the identified values in respect to their sensitivity, e.g. for the relaxation time. But for this purpose it is necessary to give a short

overview of the production process of PTFE compounds.

The thermo-mechanical properties of these materials are not only determined by their components (matrix and filler particles) but also by the way they are produced. First the granulated PTFE is mixed with the filler particles (e.g. MoS₂ and glass fibres). Then everything is compressed and sintered. Variations in pressure during compression, and temperature during the sintering process make the final mechanical properties of the PTFE compounds adjustable to specific applications. Hence, if typical values of PTFE compounds have to be discussed, the influence of the production process has to be taken into account. Considering this fact, the sensitivity of the identified model parameters on the relaxation time will be discussed. Therefore the model parameters estimated for compound S (Table 2) at 23 °C are compared to the constants identified for compound B (Table 3). The amount of the glass fibre FVF in both materials equals 5%. But the relaxation time, used for the numerical identification of the model parameters, is different.

Ten hours of relaxation have been used to identify the set of model parameters for compound S (Table 2, column 1). In order to identify the model parameters for compound B (Table 3, column 2) this time period has been reduced to 2 h. Hence the comparison of these two sets of model parameters allows a first assessment of the relaxation time sensitivity. Therefore the constants identified for compound S are understood as the reference values.

The comparison of both data records shows that all corresponding model parameters are in the same order of magnitude. Hence the characteristic of the material model will not vary with a decrease of the relaxation time.

The smallest relative deviation (+0.6%) is to be found in the variation of the elastic constant B . The model parameter η of the finite viscous flow rule, see Eq. (24), shows the largest relative deviation (–55.9%).

The influence of the relaxation time on the absolute values of the model parameters is not surprising, because with increasing relaxation time the stress response of the material approaches the equi-

Table 4

Typical orders of magnitude: finite endochronic viscoplastic model

Parameter	Order of magnitude
A [N/mm ²]	1×10^0
B	1×10^2
Y [N/mm ²]	1×10^1
C [N/mm ²]	2×10^2
η [s ⁻¹]	5×10^5
κ	2×10^0
σ_0 [N/mm ²]	1×10^0

librium state. Hence, for a better prediction of the material equilibrium, a long time relaxation process should be used for the identification procedure.

Summarizing the results of this discussion it is difficult to specify typical values of the model parameters. But it is possible to declare typical orders of magnitude for each constant. These data are listed in Table 4.

5. Concluding remarks

A finite endochronic viscoplastic material model based on material isomorphisms is presented. This approach allows to describe common inelastic phenomena (stress relaxation, strain rate dependence, plastic deformation, creep) and special effects (SDE) in the mechanical behaviour of thermoplastic materials. The model has been identified for various PTFE compounds numerically. The results of the presented investigations support the following facts.

- The temperature dependence of the model parameters in simple tension can be described by monotonous (decreasing) analytical functions.
- The model allows to predict SDE of PTFE compounds.
- The amount of fillers effects the elastic constants of the model as well as the inelastic parameters.

References

- Bertram, A., 1998. An alternative approach to finite plasticity based on material isomorphisms. *International Journal of Plasticity* 52, 353–374.

- Bertram, A., 2003. Finite thermoplasticity based on isomorphisms. *International Journal of Plasticity* 19, 2027–2050.
- Colak, U.O., 2004. A viscoplastic theory applied to proportional and non-proportional cyclic loading at small strains. *International Journal of Plasticity* 20, 1387–1401.
- Haupt, P., 1977. *Viskoelastizität und Plastizität*. Springer, Berlin.
- Haupt, P., 2000. *Continuum Mechanics and Theory of Materials*. Springer, Berlin.
- Khan, A., Zhang, H., 2001. Finite deformation of a polymer: experiments and modeling. *International Journal of Plasticity* 17, 1167–1188.
- Klitschkowski, T., Subramanian, S.P., Schomburg, U., 2000. Experimental investigation on the plastic memory effect of PTFE compounds. *Journal of Materials Processing Manufacturing Science* 9, 113–130.
- Klitschkowski, T., Schomburg, U., Bertram, A., 2001. Viskoplastische materialmodellierung am beispiel des dichtungswerkstoffes polytetrafluorethylen. *Technische Mechanik* 21, 227–241.
- Klitschkowski, T., Schomburg, U., Bertram, A., 2002. Endochronic viscoplastic material models for filled PTFE. *Mechanics of Materials* 34, 795–808.
- Klitschkowski, T., Schomburg, U., Bertram, A., 2003. An endochronic viscoplastic approach for materials with different behavior in tension and compression. In: Wolfgang G. Knauss (Editor-in-Chief), *Mechanics of Time-Dependent Materials*, in press.
- Krawietz, A., 1986. *Materialtheorie. Mathematische Beschreibung des Phanomenologischen Thermomechanischen Verhaltens*. Springer, Berlin.
- Krempel, E., 1996. A small-strain viscoplasticity theory based on overstress. In: Krausz, A.S., Krausz, K. (Eds.), *Unified Constitutive Laws of Plastic Deformation*. Academic Press, New York, pp. 281–318.
- Krempel, E., 2001. Relaxation behaviour and modelling. *International Journal of Plasticity* 17, 1419–1436.
- Krempel, E., Ho, K., 2002. Extension of the viscoplasticity based on overstress (VBO) to capture non-standard rate dependence in solids. *International Journal of Plasticity* 18, 851–872.
- Krempel, E., Khan, F., 2003. Rate (time)-dependent deformation behaviour: an overview of some properties of metals and solid polymers. *International Journal of Plasticity* 19, 1069–1095.
- Lin, R., 2003. Viscoelastic and elastic-viscoelastic-elastoplastic constitutive characterization of polymers at finite strains: theoretical and numerical aspects. Dissertation, Universität der Bundeswehr Hamburg.
- Pohl, H., 1999. Computergestützte und experimentelle untersuchungen von manschettendichtungen aus glasfaser-verstärktem PTFE-compound. Dissertation, Universität der Bundeswehr Hamburg.
- Olbrich, M., Reichert, M., Haas, W., 2000. Bin Überspannungsmodell zur beschreibung des mechanischen verhaltens des dichtungswerkstoffes polytetrafluorethylen. III. *Hamburger Dichtungstechnisches Kolloquium, TU Hamburg-Harburg*, pp. 1–12.
- Sui, H., 1998. Beitrag zur simulation von PTFE-wellendichtmanschetten in verbrennungsmotoren. Dissertation, Universität der Bundeswehr Hamburg.
- Valanis, K., 1971. A theory of viscoplasticity without a yield surface, Part I: general theory. *Archive of Mechanics* 23, 517–533.
- Wüstenhagen, G., 1995. Beitrag zur optimierung des entlasteten wellendichtrings. Dissertation, Universität Stuttgart.