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On the Ellipticity of Finite Isotropic Linear Elastic Laws

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Abstract. We consider isotropic linear elastic laws formulated in terms of generalized stresses and strains in the sense of Hill (1968). This class of elastic laws contains as special cases the stress-strain relationships based on Seth-strains (Seth, 1964) such as the St.Venant-Kirchhoff law or the Hencky law. It is shown that such laws fail to be globally elliptic. It is well known that the property of (global) polyconvexity (Ball, 1977) of the elastic strain energy density implies a (global) ellipticity. Hence, it can be concluded, that no linear elastic law formulated in terms of generalized stresses and strains is globally polyconvex.

Keywords: Ellipticity, Generalized strains, Generalized stresses, Hill-strains, Linear elastic laws, Polyconvexity, Seth-Hill-strains

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1. Introduction

Hooke's law is the geometrically and physically linear relation between the true or Cauchy stress tensor and the infinitesimal strain tensor. It is one of the oldest and, mathematically and physically, best understood constitutive relations in continuum mechanics. In the physically and geometrically linear case, the question of the existence and uniqueness of solutions to boundary value problems has been almost fully analyzed (see, e.g., Knops and Payne, 1971; Gurtin, 1972).

The geometrically nonlinear but physically linear formulation of Hooke's law is not unique in its form. This is due to the fact that there are arbitrarily many generalized stress and strain measures (Hill, 1968) which, of course, for small strains coincide. In mechanical and civil engineering only a few strain measures are commonly used such as Green's, Biot's, Hencky's, and Almansi's strains. The conjugate stress tensors of all of them are well known.

Under large strains all materials behave inelastically or elastically nonlinear. Nevertheless, for physically linear problems the question of existence and uniqueness is of great importance since there are applications with large rotations and small strains or applications where the linear behavior has to be extrapolated to the nonlinear range because of a lack of experimental data.

In this paper we exclusively consider the class of isotropic and physically linear but geometrically nonlinear elastic laws. Considerations of existence and uniqueness of solutions of boundary value problems require certain restrictions upon the elastic strain energy density. The restriction of convexity was realized as too strong, and nowadays polyconvexity is frequently required (Ball, 1977; Krawietz, 1986; Ciarlet, 1988; Šilhavý, 1997). The condition of polyconvexity became a powerful means to establish the existence of solutions of extremum problems of solids. It is well known that the property of global polyconvexity of the elastic strain energy density implies a global ellipticity. In the present work all isotropic linear finite elastic laws based on generalized strains are shown to lose ellipticity for certain deformations. Hence, it can be concluded, that all linear elastic laws formulated in terms of generalized stresses and strains are not globally polyconvex.

The outline of the paper is as follows. In Section 2 we present the basic equations for hyperelastic constitutive equations in terms of generalized stresses and strains. First some basic results from tensor analysis are recapitulated which are useful in this context. Then the generalized strains according to Seth (1964) and Hill (1968) are introduced. In order to make the paper self-contained we summarize the formulae for the determination of the generalized stress tensor which is conjugate to a given generalized strain tensor. The notion is based on the derivative of eigenvalues and eigenprojections of 2nd-order (Carlson and Hoger, 1986; Šilhavý, 1997; Padovani, 2000). We limit our considerations to Lagrangian stress and strain measures. This implies that an elastic law which is formulated in terms of these measures is a reduced form and, hence, identically fulfills the principle of objectivity.

In Section 3 the strong ellipticity condition is analyzed for hyperelastic materials which obey a physically linear stress-strain relationship. Based on the aforementioned formulae for the derivative of the eigenvalues und eigenprojections, the acoustic tensor is derived for isotropic hyperelastic materials. The case of coalescent eigenvalues is discussed. We shortly summarize a theorem by Simpson and Spector (1983) which gives necessary and sufficient conditions for the elasticity tensor to be strongly elliptic at certain values of the deformation gradient. The theorem is applied to the considered class of materials, and it is shown that a linearization of the generalized stress-strain relationship excludes a global ellipticity.

In Section 4 we analyze the size of the domain in the stretch space for which the strong ellipticity condition is fulfilled for strain energy densities based on Seth-Hill strains. For that purpose a sphere is inscribed into the stretch space centered at the undeformed placement. Then the critical radii are determined numerically for which a loss of strong ellipticity occurs. It is shown that the formulation of the elastic law based on the Hencky strain tensor has the largest elliptic domain in the sense of the aforementioned radius.

Notation. Throughout the text a direct tensor notation is preferred. To avoid additional formal definitions, the index notation is applied in some cases using the summation convention. A linear mapping of 2nd-order tensors is written as $\mathbf{A} = \mathbb{C}[\mathbf{B}]$. The components of $\mathbf{A}$ with respect to an orthonormal basis are $A_{ij} = C_{ijkl}B_{kl}$. Alternatively, we use the following mapping $\mathbf{A} = \mathbb{C}[[\mathbf{B}]]$, which is defined through $A_{ij} = C_{ikjl}B_{kl}$. The scalar product, the dyadic product, and the Frobenius norm are denoted by $\mathbf{A} \cdot \mathbf{B}$, $\mathbf{A} \otimes \mathbf{B}$, and $\|\mathbf{A}\| = \sqrt{\mathbf{A} \cdot \mathbf{A}}$, respectively. $\mathbf{I} = \mathbf{e}_i \otimes \mathbf{e}_i$ and $\mathbb{I} = \mathbf{e}_i \otimes \mathbf{e}_j \otimes (\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i)/2$ denote the 2nd-order identity and the identity on symmetric 2nd-order tensors. The transposition of a 4th-order tensor is defined by $\mathbf{A} \cdot \mathbb{C}[\mathbf{B}] = \mathbf{B} \cdot \mathbb{C}^{\mathsf{T}}[\mathbf{A}] \forall \mathbf{A}, \mathbf{B}$. Inv^+ denotes the set of invertible 2nd-order tensors with positive determinant. Sym, Sym^+, and Ort^+ represent the sets of symmetric, symmetric and positive definite, and proper orthogonal 2nd-order tensors, respectively. $Vect$ is a three-dimensional vector space. $\mathcal{R}$ and $\mathcal{R}^+$ are the set of real and positive real numbers.

2. Elastic Laws Formulated in Terms of Generalized Stresses and Strains

Spectral Decomposition of 2nd-order Tensors. The spectral decomposition of a 2nd-order tensor $\mathbf{C} \in Sym$ with different eigenvalues $c_1 > c_2 > c_3$

$$\mathbf{C} = \sum_{i=1}^{3} c_i \mathbf{P}_i, \qquad \mathbf{P}_i = \mathbf{c}_i \otimes \mathbf{c}_i \tag{1}$$

is unique. $\mathbf{P}_i$ are the eigenprojections and the $\mathbf{c}_i$ are the eigenvectors of $\mathbf{C}$. In the case of repeated eigenvalues, the eigenprojections are still unique, but the eigenvectors are not. The projectors $\mathbf{P}_i$ are generally idempotent $\mathbf{P}_i\mathbf{P}_i = \mathbf{P}_i$, biorthogonal $\mathbf{P}_i\mathbf{P}_j = \mathbf{0}$ $(i \neq j)$, and complete $\sum_{i=1}^{k} \mathbf{P}_i = \mathbf{I}$, where k denotes the number of distinct eigenvalues.

The derivatives of the eigenvalues c_i and eigenprojections $\mathbf{P}_i$ with respect to $\mathbf{C}$ are well known (Carlson and Hoger, 1986; Šilhavý, 1997; Padovani, 2000). In this section, only the case of distinct eigenvalues is considered. Later on we specify the equations for the case of coalescent eigenvalues. The derivative of the eigenvalues c_i with respect to $\mathbf{C}$ is

$$\frac{\partial c_i}{\partial \mathbf{C}} = \mathbf{P}_i. \tag{2}$$

The derivative of the eigenprojections $\mathbf{P}_i$ with respect to $\mathbf{C}$ is

$$\frac{\partial \mathbf{P}_1}{\partial \mathbf{C}} = \frac{1}{c_1 - c_2}\mathbb{P}_{12} - \frac{1}{c_3 - c_1}\mathbb{P}_{31}, \tag{3}$$

$$\frac{\partial \mathbf{P}_2}{\partial \mathbf{C}} = \frac{1}{c_2 - c_3}\mathbb{P}_{23} - \frac{1}{c_1 - c_2}\mathbb{P}_{12}, \tag{4}$$

$$\frac{\partial \mathbf{P}_3}{\partial \mathbf{C}} = \frac{1}{c_3 - c_1}\mathbb{P}_{31} - \frac{1}{c_2 - c_3}\mathbb{P}_{23}, \tag{5}$$

with

$$\begin{aligned} \mathbb{P}_{ij} \;=\; & \tfrac{1}{2}\left(\mathbf{c}_i \otimes \mathbf{c}_j \otimes \mathbf{c}_i \otimes \mathbf{c}_j + \mathbf{c}_i \otimes \mathbf{c}_j \otimes \mathbf{c}_j \otimes \mathbf{c}_i + \right. \\ & \left. \mathbf{c}_j \otimes \mathbf{c}_i \otimes \mathbf{c}_i \otimes \mathbf{c}_j + \mathbf{c}_j \otimes \mathbf{c}_i \otimes \mathbf{c}_j \otimes \mathbf{c}_i\right). \end{aligned} \tag{6}$$

A particular tensor valued tensor function, which plays an important role for the definition of generalized strains, is

$$\mathbf{f}(\mathbf{C}) = f(c_i)\mathbf{P}_i \tag{7}$$

with $f : \mathcal{R}^+ \to \mathcal{R}$. It is easy to see that the function $\mathbf{f}(\mathbf{C})$ is isotropic in the sense that

$$\mathbf{Q}\mathbf{f}(\mathbf{C})\mathbf{Q}^\mathsf{T} = \mathbf{f}(\mathbf{Q}\mathbf{C}\mathbf{Q}^\mathsf{T}) \qquad \forall \mathbf{Q} \in Ort^+. \tag{8}$$

It should be noted that $\mathbf{f}(\mathbf{C})$ represents only a special case of the general isotropic tensor function of a symmetric 2nd-order tensor because the $f_i = f(c_i)$ depend only on one eigenvalue.

Based on the formulae for the derivative of the eigenvalues and eigenprojections, the spectral decomposition of the gradient of $\mathbf{f}(\mathbf{C})$ can be determined (Xiao et al., 1998). This decomposition will allow for expressing the generalized stresses in a compact form. We consider again the case of distinct eigenvalues. Let f be a differentiable function, then its gradient is the 4th-order tensor

$$\frac{\partial \mathbf{f}(\mathbf{C})}{\partial \mathbf{C}} = \sum_{\substack{i,j=1 \\ i \geq j}}^{6} g_{ij}\mathbf{G}_{ij} \otimes \mathbf{G}_{ij} \tag{9}$$

with

$$g_{ij} = \begin{cases} f'(c_i) & \mid i = j \\ \dfrac{f(c_i) - f(c_j)}{c_i - c_j} & \mid i \neq j \end{cases} \tag{10}$$

and

$$\mathbf{G}_{ij} = \begin{cases} \mathbf{c}_i \otimes \mathbf{c}_i & \mid i = j \\ \frac{\sqrt{2}}{2}(\mathbf{c}_i \otimes \mathbf{c}_j + \mathbf{c}_j \otimes \mathbf{c}_i) & \mid i \neq j. \end{cases} \tag{11}$$

The g_{ij} are eigenvalues and the $\mathbf{G}_{ij}$ eigentensors of the gradient of $\mathbf{f}(\mathbf{C})$.

Generalized Strains. The polar decomposition of the deformation gradient $\mathbf{F} = \mathbf{RU} \in Inv^+$ will be used later on, where $\mathbf{R} \in Ort^+$ denotes the orthogonal part and $\mathbf{U} \in Sym^+$ the symmetric positive definite part of $\mathbf{F}$. The right Cauchy-Green tensor is given by $\mathbf{C} = \mathbf{F}^{\mathsf{T}}\mathbf{F} \in Sym^+$. By means of a function of the type (7) the generalized strains or Hill strains $\mathbf{E} \in Sym$ (Hill, 1968) can be defined in terms of the right Cauchy-Green tensor

$$\mathbf{E} = \mathbf{f}(\mathbf{C}) = f(c_i)\mathbf{P}_i, \tag{12}$$

where c_i and $\mathbf{P}_i$ denote the eigenvalues and eigenprojections of $\mathbf{C}$. f is assumed to be twice continuously differentiable and monotonous with the properties

$$f(1) = 0, \qquad f'(1) = \frac{1}{2}, \qquad f' \neq 0. \tag{13}$$

Equation $(13)_1$ ensures that $\mathbf{E}$ coincides with the infinitesimal strain tensor for small deformations. A special class of generalized strains, the Seth-Hill strains (Seth, 1964), are defined by the function ($m \in \mathcal{R}$)

$$f(x) = \begin{cases} \dfrac{1}{2m}(x^m - 1) & \mid m \neq 0 \\ \ln(x) & \mid m = 0. \end{cases} \tag{14}$$

All commonly used strain measures fall within this class, such as Green's strain ($m = 1$), Biot's strain ($m = 1/2$), Hencky's strain ($m = 0$), or Almansi's strain ($m = -1$).

Generalized Stresses. A generalized stress tensor $\mathbf{T}$ can be defined by the following requirement: the inner product of $\mathbf{T}$ with $\dot{\mathbf{E}}$ yields the production of internal energy per unit referential volume in the balance of internal energy (Hill, 1968, Ogden, 1984, Šilhavý, 1997). A dot denotes the material time derivative. The production term is of the form

$$\boldsymbol{\tau} \cdot \mathbf{D} = \frac{1}{2}\mathbf{S} \cdot \dot{\mathbf{C}} = \mathbf{T} \cdot \dot{\mathbf{E}}, \tag{15}$$

where $\boldsymbol{\tau} \in Sym$, $\mathbf{S} = \mathbf{F}^{-1}\boldsymbol{\tau}\mathbf{F}^{-\mathsf{T}} \in Sym$, and $\mathbf{D} \in Sym$ are the Kirchhoff stress, the 2.Piola-Kirchhoff stress, and the rate of deformation, respectively. The relation between $\mathbf{S}$ and $\mathbf{T}$

$$\mathbf{S} = 2\left(\frac{\partial \mathbf{f}(\mathbf{C})}{\partial \mathbf{C}}\right)^{\mathsf{T}}[\mathbf{T}] \tag{16}$$

is obtained by (15) by applying the chain rule. The inverse relation is

$$\mathbf{T} = \frac{1}{2}\left(\frac{\partial \mathbf{f}(\mathbf{C})}{\partial \mathbf{C}}\right)^{-\mathsf{T}} [\mathbf{F}^{-1}\boldsymbol{\tau}\mathbf{F}^{-\mathsf{T}}], \tag{17}$$

where the inverse of the 4th-order tensor $\partial \mathbf{f}(\mathbf{C})/\partial \mathbf{C}$ can be computed using the spectral decomposition (9). By applying equations (9)-(11), we obtain

$$\mathbf{S} = 2 \sum_{\substack{i,j=1 \\ i \geq j}}^{6} g_{ij} T_{ij} \mathbf{G}_{ij}, \tag{18}$$

where the T_{ij} are the components of $\mathbf{T}$ with respect to $\mathbf{G}_{ij}$. For the special case of elastic isotropy, it can be easily shown that $\mathbf{T}$ and $\mathbf{E}$ are coaxial, i.e. both tensors have identical eigenprojections. As a result, the last equation simplifies to

$$\mathbf{S} = 2 \sum_{i=1}^{3} g_{ii} T_{ii} \mathbf{G}_{ii} = 2 \sum_{i=1}^{3} f'(c_i) t_i \mathbf{P}_i, \qquad \mathbf{T} = t_i \mathbf{P}_i. \tag{19}$$

If the Seth-Hill strains are considered, the g_{ij} are

$$g_{ij} = \begin{cases} \dfrac{1}{2} c_i^{m-1} & | \ i = j \\[2ex] \dfrac{1}{2m} \dfrac{c_i^m - c_j^m}{c_i - c_j} & | \ i \neq j. \end{cases} \tag{20}$$

Hence, an isotropy of the elastic law implies the following simple relation

$$\mathbf{S} = \sum_{i=1}^{3} c_i^{m-1} t_i \mathbf{P}_i = \mathbf{T}\mathbf{C}^{m-1} = \mathbf{T}\mathbf{U}^{2(m-1)}. \tag{21}$$

It should be remarked that the generalized stresses are unique also for repeated eigenvalues since

$$\lim_{c_j \to c_i} \frac{f(c_i) - f(c_j)}{c_i - c_j} = f'(c_i). \tag{22}$$

From equations (10), (18), and (22) it is clear that for infinitesimal deformations the generalized stresses coincide with the Cauchy stress.

Elastic Materials. For an elastic material the stresses are a function of the current deformation (Truesdell and Noll, 1965). Hence, the equation

$$\mathbf{T} = \mathbf{k}(\mathbf{E}) \tag{23}$$

is the general constitutive equation of an elastic material. Since $\mathbf{T}$ and $\mathbf{E}$ are independent of the observer, (23) represents a reduced form of an elastic law. The material symmetry can be defined by its symmetry group S with respect to the undistorted state. The elements of S satisfy

$$\mathbf{Q}\mathbf{k}(\mathbf{E})\mathbf{Q}^{\mathsf{T}} = \mathbf{k}(\mathbf{Q}\mathbf{E}\mathbf{Q}^{\mathsf{T}}) \qquad \forall \mathbf{Q} \in S \subseteq Ort^{+}. \tag{24}$$

If $S = Ort^+$, then the elastic behavior is isotropic. If the function $\mathbf{k}$ can be represented by a gradient of a strain energy density W, such that

$$\mathbf{T} = \frac{\partial W(\mathbf{E})}{\partial \mathbf{E}} \tag{25}$$

holds for all strains, then the elastic behavior is called hyperelastic. In this case all paths in the strain space are dissipation free. The symmetry of an hyperelastic elastic law is specified by

$$W(\mathbf{E}) = W(\mathbf{QEQ}^\mathrm{T}) \qquad \forall \mathbf{Q} \in S \subseteq Ort^+. \tag{26}$$

In the context of physically linear and isotropic elasticity, the stress-strain relationship is given by

$$\mathbf{T} = \mathbb{C}[\mathbf{E}], \qquad \mathbb{C} = 3K\mathbb{P}_1^I + 2G\mathbb{P}_2^I, \tag{27}$$

or

$$\mathbf{T} = K\mathrm{tr}(\mathbf{E})\mathbf{I} + 2G\mathbf{E}', \qquad \mathbf{E}' = \mathbf{E} - \frac{1}{3}\mathrm{tr}(\mathbf{E})\mathbf{I}, \tag{28}$$

where $\mathbb{P}_1^I$ and $\mathbb{P}_2^I$ are the eigenprojections of $\mathbb{C}$

$$\mathbb{P}_1^I = \tfrac{1}{3}\mathbf{I} \otimes \mathbf{I}, \qquad \mathbb{P}_2^I = \mathbb{I} - \mathbb{P}_1^I. \tag{29}$$

In the small strain range, K and G are interpreted as bulk and shear modulus, respectively.

In the linear and isotropic case, elastic laws are always hyperelastic. The elastic energy W is then given by

$$\begin{aligned} W(\mathbf{E}) &= \tfrac{1}{2}\mathbf{E} \cdot \mathbb{C}[\mathbf{E}] \\ &= \tfrac{1}{2}k_1 \left(f_1^2 + f_2^2 + f_3^2\right) + k_2 \left(f_1 f_2 + f_2 f_3 + f_3 f_1\right) \end{aligned} \tag{30}$$

with

$$f_i = f(c_i), \qquad k_1 = K + \frac{4}{3}G, \qquad k_2 = K - \frac{2}{3}G. \tag{31}$$

The first and second derivative of the strain energy density with respect to the eigenvalues of $\mathbf{C}$ are

$$\frac{\partial W}{\partial c_i} = f_i' \left(k_1 f_i + k_2 \left(f_{i+1} + f_{i+2}\right)\right) \tag{32}$$

and

$$\frac{\partial^2 W}{\partial c_i \partial c_j} = \begin{cases} k_1(f_i'^2 + f_i f_i'') + k_2 f_i'' \left(f_{i+1} + f_{i+2}\right) & \mid \; i = j \\ k_2 f_i' f_j' & \mid \; i \neq j \end{cases} \tag{33}$$

with $i, j = 1, 2, 3$ modulo 3 and

$$f_i' = f'(c_i), \qquad f_i'' = f''(c_i). \tag{34}$$

3. Legendre-Hadamard Ellipticity of Linear Isotropic Elastic Laws

Hypotheses on the Material Response. There are certain natural requirements which the strain energy density W should meet (Šilhavý, 1997, Subsections 11.3 and 17.5):

- $W(\mathbf{F}) \geq 0 \; \forall \mathbf{F} \in Inv^+$ $\qquad N1,$
- $W(\mathbf{F}) \to \infty$ as $\det(\mathbf{F}) \to 0$ or $\|\mathbf{F}\| \to \infty$ $\qquad N2,$
- $W(\mathbf{F})$ is polyconvex (Ball, 1977) $\qquad N3.$

The function W is said to be polyconvex if it is polyconvex at every point of its domain, i.e., for all $\mathbf{F} \in Inv^+$ (global polyconvexity). The condition of polyconvexity can be applied to fluids and solids. It is a powerful means to prove the existence of solutions of extremum problems of solids. Global polyconvexity implies global Legendre-Hadamard ellipticity which is defined in the next paragraph.

The Legendre-Hadamard Condition. An elastic energy W is called elliptic at $\mathbf{F}$ if it satisfies the Legendre-Hadamard condition

$$\frac{\partial^2 W}{\partial \mathbf{F}^2}[\mathbf{H}, \mathbf{H}] \geq 0 \qquad \forall \mathbf{H} = \mathbf{m} \otimes \mathbf{n}, \qquad \mathbf{m}, \mathbf{n} \in Vect. \tag{35}$$

The function W is said to be elliptic if it is elliptic at every point of its domain (global ellipticity). The Legendre-Hadamard condition is a necessary condition for the infinitesimal stability of an elastic body under boundary conditions of place and traction (Truesdell and Noll, 1965, Sect. 68bis). If the placement of a body is infinitesimally stable, then for any given direction of propagation of an acceleration wave, the wave speeds are always real (Hadamard's theorem on waves). The propagation of infinitesimal plane waves in an homogeneously deformed body follows the same rules as those of the acceleration waves (Truesdell and Noll, 1965, Sect. 73). The ellipticity condition has been analyzed for two-dimensional deformations in incompressible materials by Abeyaratne (1980) and in compressible materials by Knowles and Sternberg (1977). Three-dimensional deformations have been considered by Zee and Sternberg (1983) for incompressible materials and by Simpson and Spector (1983), Ogden (1984), and Rosakis (1990) for compressible materials.

The Legendre-Hadamard condition is equivalent to the assumption that the acoustic tensor

$$\mathbf{A}(\mathbf{F}, \mathbf{n}) = \frac{1}{\varrho_0} \frac{\partial^2 W}{\partial \mathbf{F}^2}[[\mathbf{n} \otimes \mathbf{n}]] \tag{36}$$

is positive semidefinite. The strong ellipticity condition

$$\frac{\partial^2 W}{\partial \mathbf{F}^2}[\mathbf{H}, \mathbf{H}] > 0 \qquad \forall \mathbf{H} = \mathbf{m} \otimes \mathbf{n} \neq 0, \qquad \mathbf{m}, \mathbf{n} \in Vect \tag{37}$$

implies a positive definite acoustic tensor and excludes wave speeds equal to zero. If the acoustic tensor is not positive semidefinite, then there exist infinitesimal disturbances in an infinite medium which amplify with time (Hayes and Rivlin, 1961; Truesdell and Noll, 1965, Sect. 73). "This does not mean, necessarily, we should impose as a condition on the constitutive equation the requirement that (the acoustic tensor, T.B.&A.B.) $\mathbf{Q}(\mathbf{n})$ be such as to have only positive proper numbers in all states of strain. Rather, it indicates that for a given material any homogeneous strain giving $\mathbf{Q}(\mathbf{n})$ a negative or complex proper number is not likely to be encountered in praxis, since in such a strain there are certain kinds of disturbances which, however small in magnitude initially and in whatever way excited, begin to grow at once and hence to destroy the given state of strain" (Truesdell and Noll, 1965, Sect. 73).

The Acoustic Tensor for Isotropic Materials. The acoustic tensor for isotropic nonlinear-elastic materials has been given explicitly in terms of the singular values of $\mathbf{F}$ and the first and second derivatives of the elastic energy, e.g., by Ogden (1984) and Šilhavý (1997). Based on the formulae for the derivative of the eigenvalues c_i und eigenprojections $\mathbf{P}_i$ with respect to $\mathbf{C}$, the acoustic tensor can be derived in a straightforward manner by applying the chain rule. One obtains from equation (36)

$$\begin{aligned} \varrho_0 \mathbf{A} &= \textstyle\sum_{i,j=1}^3 4W_{,ij}(\mathbf{F}\mathbf{P}_i\mathbf{n}) \otimes (\mathbf{F}\mathbf{P}_j\mathbf{n}) \\ &+ \textstyle\sum_{i=1}^3 2W_{,i}\,(\mathbf{n}\cdot\mathbf{P}_i\mathbf{n})\,\mathbf{I} \\ &+ \textstyle\sum_{i=1}^3 4W_{,i}\mathbf{F}\left(\mathbf{n}\frac{\partial \mathbf{P}_i}{\partial \mathbf{C}}\mathbf{n}\right)\mathbf{F}^{\mathrm{T}} \end{aligned} \tag{38}$$

with

$$W_{,i} = \frac{\partial W}{\partial c_i} \qquad W_{,ij} = \frac{\partial^2 W}{\partial c_i \partial c_j}. \tag{39}$$

Eliminating the derivative $\partial \mathbf{P}_i / \partial \mathbf{C}$ by (3) - (5) and applying the polar decomposition of $\mathbf{F}$ gives

$$\varrho_0 \mathbf{A} = \mathbf{R}\left(\sum_{i=j} B_{iij} n_j^2 \mathbf{c}_i \otimes \mathbf{c}_i + \sum_{i\neq j} B_{ij} n_i n_j \mathbf{c}_i \otimes \mathbf{c}_j\right)\mathbf{R}^{\mathrm{T}}, \tag{40}$$

where

$$B_{iij} = B_{jji} = \begin{cases} 2\left(2c_i W_{,ii} + W_{,i}\right) & \mid\ i = j \\[2ex] 2\dfrac{c_i W_{,i} - c_j W_{,j}}{c_i - c_j} & \mid\ i \neq j \end{cases} \tag{41}$$

and

$$B_{ij} = B_{ji} = 2\sqrt{c_i c_j}\left(2W_{,ij} + \frac{W_{,i} - W_{,j}}{c_i - c_j}\right). \tag{42}$$

Equation (40) shows the implication of the principle of objectivity for the acoustic tensor

$$W(\mathbf{F}) = W(\mathbf{Q}\mathbf{F})\ \forall \mathbf{Q} \in \mathit{Ort}^+ \quad \Rightarrow \quad \mathbf{A}(\mathbf{F}, \mathbf{n}) = \mathbf{R}\mathbf{A}(\mathbf{U}, \mathbf{n})\mathbf{R}^{\mathrm{T}}. \tag{43}$$

It can be concluded that the orthogonal part $\mathbf{R}$ of $\mathbf{F}$ does not affect the definiteness of the acoustic tensor. Hence, $\mathbf{R} = \mathbf{I}$ can be assumed in the subsequent considerations.

Up to now only the case of distinct eigenvalues has been considered. Now let us assume that two or three eigenvalues coincide. In this case the B_{iij} and the B_{ij} are determined by a limiting process. One obtains

$$\lim_{c_i \to c_j} B_{iij} = \begin{cases} 2c_i\left(2c_i W_{,ii} + W_{,i}\right) & \mid\ i = j \\[2ex] 2\left(W_{,i} + c_i\left(W_{,ii} - W_{,ij}\right)\right) & \mid\ i \neq j \end{cases} \tag{44}$$

and

$$\lim_{c_i \to c_j} B_{ij} = 2c_i \left(W_{,ij} + W_{,ii} \right). \tag{45}$$

If the right stretch tensor $\mathbf{U} = \lambda_i \mathbf{P}_i = \sqrt{c_i}\,\mathbf{P}_i$ is used instead of $\mathbf{C}$ then

$$B_{iij} = B_{jji} = \begin{cases} \dfrac{\partial^2 W}{\partial \lambda_i^2} & \mid i = j \\[2ex] \dfrac{\lambda_i \dfrac{\partial W}{\partial \lambda_i} - \lambda_j \dfrac{\partial W}{\partial \lambda_j}}{\lambda_i^2 - \lambda_j^2} & \mid i \neq j \end{cases} \tag{46}$$

and

$$B_{ij} = B_{ji} = \frac{\partial^2 W}{\partial \lambda_i \partial \lambda_j} + \frac{\lambda_j \dfrac{\partial W}{\partial \lambda_i} - \lambda_i \dfrac{\partial W}{\partial \lambda_j}}{\lambda_i^2 - \lambda_j^2}. \tag{47}$$

The Strong Ellipticity Condition for Isotropic Materials. The elasticity tensor is strongly elliptic at $\mathbf{F} \in Inv^+$ if and only if the acoustic tensor $\mathbf{A}(\mathbf{F}, \mathbf{n}) \in Sym$ is positive definite for all directions $\mathbf{n}$ (Truesdell and Noll, 1965). The quadratic form $\mathbf{m} \cdot \varrho_0 \mathbf{A}(\mathbf{F}, \mathbf{n})\mathbf{m}$ can be written in the form of

$$\begin{aligned} \mathbf{m} \cdot \varrho_0 \mathbf{A}(\mathbf{F}, \mathbf{n})\mathbf{m} &= B_{112} \left(H_{12} - \delta_1 \delta_2 H_{21} \right)^2 \\ &+ B_{113} \left(H_{13} - \delta_1 \delta_3 H_{31} \right)^2 \\ &+ B_{223} \left(H_{23} - \delta_2 \delta_3 H_{32} \right)^2 \\ + \begin{bmatrix} \delta_1 H_{11} \\ \delta_2 H_{22} \\ \delta_3 H_{33} \end{bmatrix} & \underbrace{\begin{bmatrix} B_{111} & B_{112} + \delta_1 \delta_2 B_{12} & B_{113} + \delta_1 \delta_3 B_{13} \\ \cdot & B_{222} & B_{223} + \delta_2 \delta_3 B_{23} \\ \cdot & \cdot & B_{333} \end{bmatrix}}_{Y_{ij}} \begin{bmatrix} \delta_1 H_{11} \\ \delta_2 H_{22} \\ \delta_3 H_{33} \end{bmatrix}. \end{aligned} \tag{48}$$

Necessary and sufficient conditions for the elasticity tensor to be strongly elliptic at $\mathbf{F} \in Inv^+$ have been given by Simpson and Spector (1983): $B_{112} > 0$, $B_{113} > 0$, $B_{223} > 0$, and the matrix Y_{ij} strictly copositive for all eight choices $\delta_1, \delta_2, \delta_3 = \pm 1$. An $n \times n$ matrix Y_{ij} is said to be strictly copositive (Cottle et al., 1970) if

$$x_i Y_{ij} x_j > 0 \qquad \forall x_i \geq 0 \quad \text{with} \quad \sum_{i=1}^{n} x_i \neq 0. \tag{49}$$

Applying the Theorem 4.2 by Simpson and Spector (1983) on copositive matrices, necessary and sufficient conditions for the elasticity tensor to be strongly elliptic at $\mathbf{F} \in Inv^+$ are obtained in the form

$$B_{111} > 0, \qquad B_{222} > 0, \qquad B_{333} > 0, \tag{I1...I3}$$

$$B_{112} > 0, \qquad B_{113} > 0, \qquad B_{223} > 0, \tag{I4...I6}$$

$$K = B_{112} + \sqrt{B_{111}B_{222}} > |\, B_{12} \,|, \tag{I7}$$

$$L = B_{113} + \sqrt{B_{111}B_{333}} > |\, B_{13} \,|, \tag{I8}$$

$$M = B_{223} + \sqrt{B_{222}B_{333}} > |\, B_{23} \,|, \tag{I9}$$

$$\begin{aligned} &\sqrt{B_{111}B_{222}B_{333}} + \sqrt{B_{333}\,(B_{112} + \delta_1 B_{12})} \;+ \\ &\sqrt{B_{222}\,(B_{113} + \delta_2 B_{13})} + \sqrt{B_{111}\,(B_{223} + \delta_1\delta_2 B_{23})} \;+ \\ &\sqrt{2\,(L + \delta_1 B_{12})\,(M + \delta_2 B_{13})\,(N + \delta_1\delta_2 B_{23})} \;> 0 \end{aligned} \tag{I10...I13}$$

for all four choices $\delta_1, \delta_2 = \pm 1$.

Implication of the Requirement $N2$**.** Now the implications of the aforementioned requirements on W according to (30) are analyzed. If the dilatation $\mathbf{C} = c\mathbf{I}$ is considered, one has $c = \det(\mathbf{F})^{\frac{2}{3}}$ and $c = \frac{1}{3}\|\mathbf{F}\|^2$. The strain energy density is given by

$$W = \frac{9}{2} K f(c)^2. \tag{50}$$

$N2$ implies that

$$|\, f(x) \,| \to \infty \qquad \text{as} \qquad x \to 0 \qquad \text{or} \qquad x \to \infty. \tag{51}$$

It can be concluded that the Hencky strain ($m = 0$) is the only admissible strain within the Seth-Hill strains meeting this requirement.

The Strong Ellipticity Condition with Respect to the Reference Placement. If the B_{iij} and the B_{ij} are determined from (44) and (45) for the case $\mathbf{C} = \mathbf{I}$, one obtains

$$B_{111} = B_{222} = B_{333} = k_1 = K + \frac{4}{3} G, \tag{52}$$

$$B_{112} = B_{113} = B_{223} = \frac{1}{2}(k_1 - k_2) = G, \tag{53}$$

$$B_{12} = B_{13} = B_{13} = \frac{1}{2}(k_1 + k_2) = K + \frac{1}{3} G. \tag{54}$$

Exploitation of the inequalities $I4$ to $I9$ gives $G > 0$ whereas the inequalities $I1$ to $I3$ and $I10$ to $I13$ imply $3K + 4G > 0$. These are the classical results from the geometrically linear theory of elasticity. For all Hill strains the elasticity tensor is strongly elliptic at $\mathbf{F} = \mathbf{I}$ if and only if the following inequality hold: $G > 0$ and $3K + 4G = \lambda + 2G > 0$. The strong ellipticity condition at $\mathbf{F} = \mathbf{I}$ does not ensure that the bulk modulus is positive. If the tensor $\partial\mathbf{T}/\partial\mathbf{E}$ is positive definite, i.e. $K > 0$ and $G > 0$, then for all Hill strains the elasticity tensor is strongly elliptic at $\mathbf{F} = \mathbf{I}$. The aforementioned implications have been expected to hold also in the case of a physically linear law formulated in generalized stress and strain measures, because these measures are defined in a way to coincide as a limit with the stress and strain measures in the geometrically linear theory of elasticity.

Physically Linear Elastic Laws. In this paragraph the main result of this article is presented, which answers the following question: Is there a generalized strain $\mathbf{E}$ such that a finite isotropic linear elastic law (27) is strongly elliptic for all $\mathbf{F} \in Inv^+$? For some of the elastic laws given as special cases of (27) the question can already be answered. The St. Venant-Kirchhoff material ($m = 1$) is not globally elliptic and hence not globally polyconvex (Krawietz, 1986; Raoult, 1986; Ciarlet, 1988). Furthermore, the Hencky material ($m = 0$) is not globally elliptic and hence not globally polyconvex (Neff, 2000).

For the uniaxial stretch the right Cauchy-Green tensor is

$$\mathbf{C} = \mathbf{I} + (c_1 - 1)\mathbf{e}_1 \otimes \mathbf{e}_1. \tag{55}$$

Exploiting inequality $I2$, the critical stretch c_{crit} can be computed such that the strong ellipticity condition is violated

$$B_{222} = 0 \qquad \Rightarrow \qquad c_{crit} = f^{-1}\left(\frac{-k_1}{4k_2\left(f''(1) + \frac{1}{4}\right)}\right). \tag{56}$$

A similar computation with inequality $I6$ results in

$$B_{223} = 0 \qquad \Rightarrow \qquad c_{crit} = f^{-1}\left(\frac{-(k_1 - k_2)}{4k_2\left(f''(1) + \frac{1}{2}\right)}\right), \tag{57}$$

where $f^{-1} : (-\infty, \infty) \rightarrow \mathcal{R}^+$. The second derivative f'' at $c_i = 1$ is generally unknown for the Hill strains. If $f''(1) = -1/4$ then (57) gives a critical stretch, if $f''(1) = -1/2$ then (56) can be used to determine c_{crit}, and otherwise both equations allow the determination of a c_{crit}. It is easy to see that $\mathrm{d}B_{222}/\,\mathrm{d}c_1 \neq 0 \forall c_1$ if $f''(1) \neq -1/4$ and that $\mathrm{d}B_{223}/\,\mathrm{d}c_1 \neq 0 \forall c_1$ if $f''(1) \neq -1/2$. Inspections of (48) shows, that

$$\begin{aligned} \mathbf{m} = \mathbf{e}_2, \quad \mathbf{n} = \mathbf{e}_3 \quad &\Rightarrow \quad \mathbf{m} \cdot \varrho_0 \mathbf{A}(\mathbf{F}, \mathbf{n})\mathbf{m} = B_{223}, \\ \mathbf{m} = \mathbf{e}_2, \quad \mathbf{n} = \mathbf{e}_2 \quad &\Rightarrow \quad \mathbf{m} \cdot \varrho_0 \mathbf{A}(\mathbf{F}, \mathbf{n})\mathbf{m} = B_{222}. \end{aligned} \tag{58}$$

If c_1 passes c_{crit} then the acoustic tensor is no longer positive semidefinite. Hence, there exists uniaxial stretch such that an elastic energy W, which is a quadratic form of $\mathbf{E}$, does not fulfill the Legendre-Hadamard condition (35). Finally it can be concluded that: There does not exist a generalized strain tensor $\mathbf{E}$ such that an isotropic linear elastic law $\mathbf{T} = K\mathrm{tr}(\mathbf{E})\mathbf{I} + 2G\mathbf{E}'$ is elliptic at all $\mathbf{F} \in Inv^+$. It is well known that the property of (global) polyconvexity (Ball, 1977) of the elastic strain energy density implies a (global) ellipticity. Hence, there no physically linear elastic law formulated in terms of generalized stresses and strains that is globally polyconvex.

4. Domains of Ellipticity

In this section we want to estimate the size of the domain of strong ellipticity of physically linear elastic laws based on Seth-Hill strains. For an energy being quadratic in the Hencky strain tensor, there is a domain in the strain space with the elasticity tensor being strongly elliptic, which is independent of the elastic constants. The range is given by the cube $c_i \in [0.0448, 1.9477]$ which has been determined analytically by Bruhns et al. (2001). An estimate of the domain of ellipticity of a physically nonlinear elastic material (Blatz-Ko special II) can be found in Knowles and Sternberg (1975).

In the subsequent consideration we aim to estimate the size of the range in the stretch space with an elliptic elasticity tensor. For different m in (14) the inequalities $I1 \dots I13$ are checked numerically using the dimensionless energy

$$W^* = \frac{W}{k_1} = \frac{1}{2}\left(f_1^2 + f_2^2 + f_3^2\right) + \beta\left(f_1 f_2 + f_2 f_3 + f_3 f_1\right), \tag{59}$$

where

$$\beta = \frac{k_2}{k_1} = \frac{1-\alpha}{1+2\alpha}, \qquad \alpha = \frac{2G}{3K}. \tag{60}$$

It is assumed that

$$K > 0, \qquad G > 0, \tag{61}$$

which implies $\alpha > 0$ and guarantees that the reference placement is elliptic for all Seth-Hill strains. Furthermore, the Poisson's ratio ν is assumed to be positive

$$\nu = \frac{1-\alpha}{2+\alpha} > 0. \tag{62}$$

The equations (61) and (62) imply that

$$\alpha \in (0,1) \qquad \Rightarrow \qquad \beta = \frac{1-\alpha}{1+2\alpha} \in (0,1). \tag{63}$$

The positivity of ν implies the positivity of the first Lamé constant $\lambda = K - 2G/3 > 0$.

In order to determine numerically the range of stretches with an elliptic elasticity tensor we apply the following procedure. We inscribe a sphere into the three-dimensional space of eigenvalues c_i with the center at $\mathbf{C} = \mathbf{I}$ and with the radius R. The radius is then increased from 0 up to a critical value, for which somewhere on the sphere the strong ellipticity condition is violated. The radius R quantifies the amount of deformation for which the strong ellipticity condition is violated. It is given by $R = \|\mathbf{C} - \mathbf{I}\| = 2\|\mathbf{E}^G\|$ where $\mathbf{E}^G$ denotes the Green strain tensor. A point in the space of eigenvalues c_i of $\mathbf{C}$ can be described by the vector $\vec{\mathbf{c}} = c_i \mathbf{e}_i$ with $\{\mathbf{e}_i\}$ an orthonormal basis in *Vect*. In order to parameterize the surface of the sphere, the spherical coordinates

$$u_1 = R\cos(\varphi)\sin(\theta), \quad u_2 = R\sin(\varphi)\sin(\theta), \quad u_3 = R\cos(\theta) \tag{64}$$

are introduced with respect to the orthonormal basis $\{\tilde{\mathbf{e}}_i\}$ which is related to $\{\mathbf{e}_i\}$ by

$$\begin{aligned} \tilde{\mathbf{e}}_1 &= \tfrac{\sqrt{2}}{2}\left(\mathbf{e}_1 - \mathbf{e}_2\right), \\ \tilde{\mathbf{e}}_2 &= \tfrac{\sqrt{6}}{6}\left(\mathbf{e}_1 + \mathbf{e}_2 - 2\mathbf{e}_3\right), \\ \tilde{\mathbf{e}}_3 &= \tfrac{\sqrt{3}}{3}\left(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3\right). \end{aligned} \tag{65}$$

A point in the stretch space can now be expressed by

$$\vec{\mathbf{c}} = \sum_{i=1}^{3} \mathbf{e}_i + u_i \tilde{\mathbf{e}}_i. \tag{66}$$

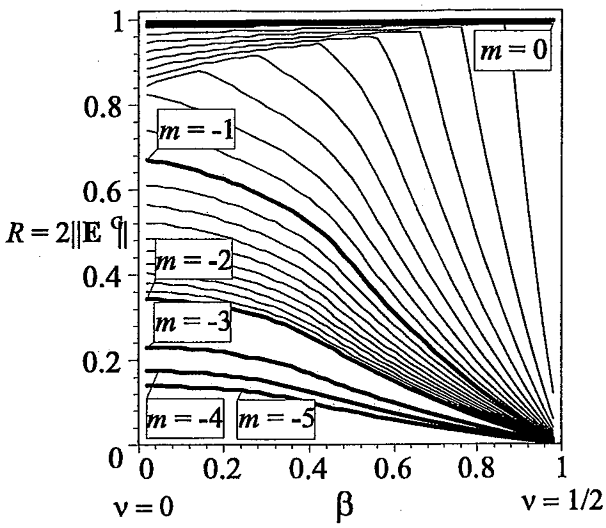

Figure 1. Range of ellipticity for negative m: R vs β

The advantage of the basis $\{\tilde{e}_i\}$ is that special cases of coinciding eigenvalues, which require a modified computation of components of the acoustic tensor, can be simply expressed by φ and θ as

$$\begin{aligned}
c_1 = c_2 = c_3 &: \theta = 0°, 180°, \\
c_1 = c_2 &: \varphi = 90°, 270°, \\
c_1 = c_3 &: \varphi = 150°, 330°, \\
c_2 = c_3 &: \varphi = 30°, 210°.
\end{aligned} \tag{67}$$

In Figures 1 and 2 the critical radii are shown for different values m and β. It can be seen that only in the case of the Hencky strain ($m = 0$) there is a finite range which does not depend on β. Furthermore, for $m = 0$ the deformation range is the largest one for all β compared to other Seth-Hill strains. If the magnitude of m is increased the ellipticity range is shrinking. For all $m \neq 0$ the range with an elliptic elasticity tensor is arbitrarily small if β is close to 1/2 ($\nu = 1/2$).

5. Conclusions

All applications in hyperelasticity are based on Seth-Hill strains. If an isotropic elastic stress-strain relationship is physically linearized in the Seth-Hill strains, then the strain energy density does not globally fulfill the ellipticity condition and hence is not globally polyconvex. This statement holds also for the more general class of Hill strains. Hence, if a polyconvex strain energy density is required in the isotropic hyperelastic case, then the stress-strain relationship must be physically non-linear in the generalized strain. For all Seth-Hill strains with $m \neq 0$ the deformation range for strong ellipticity is arbitrarily small if the Poisson ratio tends to 1/2. Only a linearization in Hencky's strain tensor results in an elastic law with a finite deformation range of a strong ellipticity for all material parameters $\lambda > 0$ and $G > 0$.

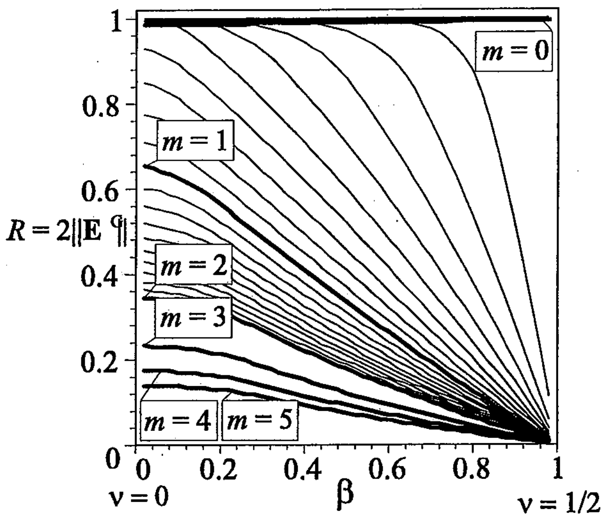

Figure 2. Range of ellipticity for positive m: R vs β

References

Abeyaratne, R.: 1980, 'Discontinous deformations gradients in plane finite elastostatics of incompressible materials'. *J. Elast.* **10**, 255–293.

Ball, J.: 1977, 'Convexity conditions and existence theorems in nonlinear elasticity'. *Arch. Rat. Mech. Anal.* **63**, 337–403.

Bruhns, O., H. Xiao, and A. Meyers: 2001, 'Constitutive inequalities for an isotropic elastic strain-energy function based on Hencky's logarithmic tensor'. *Proc. R. Soc. Lond. A* **457**, 2207–2226.

Carlson, D. and A. Hoger: 1986, 'The derivative of a tensor-valued function of a tensor'. *Quart. Appl. Math.* **XLIV**(3), 409–423.

Ciarlet, P.: 1988, *Mathematical Elasticity: Three-dimensional elasticity*, Vol. I. Elsevier, Amsterdam.

Cottle, R., G. Habetler, and C. Lemke: 1970, 'On classes of copositive matrices'. *Lin. Alg. Appl.* **3**, 295–310.

Gurtin, M.: 1972, *The Linear Theory of Elasticity*, Vol. VIa/2 of *Encyclopedia of Physics*. Springer.

Hayes, M. and R. Rivlin: 1961, 'Propagation of a plane wave in an Isotropic elastic material subjected to pure homogeneous deformation'. *Arch. Rat. Mech. Anal.* **8**(1), 15–22.

Hill, R.: 1968, 'On constitutive inequalities for simple materials'. *J. Mech. Phys. Solids* **16**, 229–242.

Knops, R. and L. Payne: 1971, *Uniquenesss theorems in linear elasticity*, Vol. 19 of *Springer Tracts in Natural Philosophy*. Springer.

Knowles, J. and E. Sternberg: 1975, 'On the ellipticity of the equations of nonlinear elastostatics for a special material'. *J. Elast.* **5**(3–4), 341–361.

Knowles, J. and E. Sternberg: 1977, 'On the failure of ellipticity of the equations for finite elastostatic plane strain'. *Arch. Rat. Mech. Analysis* **63**, 321–336.

Krawietz, A.: 1986, *Materialtheorie.* Springer-Verlag.

Neff, P.: 2000, *Mathematische Analyse multiplikativer Elastoplastizität*, Berichte aus der Mathematik. Shaker Verlag.

Ogden, R.: 1984, *Non-linear Elastic Deformations.* Dover Publications, Inc., Mineola, New York.

Padovani, C.: 2000, 'On the derivative of some tensor-valued functions'. *J. Elast.* **58**(3), 257–268.

Raoult, A.: 1986, 'Non-polyconvexity of the stored energy function of a Saint Venant-Kirchhoff material'. *Aplikace Matematiky* **6**, 417–419.

Rosakis, P.: 1990, 'Ellipticity and deformations with discontinuous gradients in finite elastostatics'. *Arch. Rat. Mech. Analysis* **109**, 1–37.

Seth, B.: 1964, 'Generalized strain measure with application to physical problems'. In: M. Reiner and D. Abir (eds.): *Second-order effects in elasticity, plasticity, and fluid dynamics.* Pergamon Press, Oxford, pp. 162–172.

Simpson, H. and J. Spector: 1983, 'On copositive matrices and strong ellipticity for isotropic elastic materials'. *Arch. Rat. Mech. Analysis* **84**, 55–68.

Truesdell, C. and W. Noll: 1965, *The Non-Linear Field Theories of Mechanics*, Vol. III/3 of *Encyclopedia of Physics.* Springer.

Šilhavý, M.: 1997, *The Mechanics and Thermodynamics of Continuous Media.* Springer.

Xiao, H., O. Bruhns, and A. Meyers: 1998, 'Strain rates and material spins'. *J. Elast.* **52**, 1–41.

Zee, L. and E. Sternberg: 1983, 'Ordinary and strong ellipticity in the equilibrium theory of incompressible hyperelastic solids'. *Arch. Rat. Mech. Analysis* **83**, 53–90.

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