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Asymptotic values of elastic anisotropy in polycrystalline copper for uniaxial tension and compression

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Abstract

The asymptotic values of elastic anisotropy, induced in OFHC copper by uniaxial tension and compression, are determined by a phenomenological model and compared with the predictions of Taylor type texture simulations. The evolution equation for the texture-dependent anisotropic part of the effective elasticity tensor consists of two parts. The first term depends only on inelastic rate of deformation and represents the driving term for the evolving anisotropy. The second term governs the saturation of anisotropy and is linear in the anisotropic portion of the effective elasticity tensor. In the present paper it is shown that such an evolution equation implies a reasonable prediction of the asymptotic elastic anisotropy for uniaxial tension and uniaxial compression.

Keywords: Anisotropic material; Constitutive behavior; Crystal plasticity; Elastic–viscoplastic material; Finite strain; Polycrystalline material

1. Introduction

Materials consisting of single crystals with a strong elastic anisotropy, such as copper, show evolving elastic properties when submitted to large inelastic deformations. E.g., in a sheet of cold rolled copper with a thickness reduction of 95%, the maximum variation of Young's modulus equals about 30% in the cold-rolled case and about 66% in the recrystallized case [19]. It has been shown that in sheet materials a statistical correlation can be detected between the elastic and plastic properties. Stickels and Mould [18] have shown that the angular variation of Young's modulus can be used to characterize empirically the formability of the sheet with respect to both the deep drawability and the earing behavior. Therefore, the anisotropy of the (visco) plastic behavior can be inferred not only from destructive tests but also from non-destructive measurements of the elastic anisotropy.

In the present paper we consider the fourth-order coefficient of a tensorial Fourier expansion of the crystal orientation distribution function [15]. This coefficient governs the effective elastic properties of the polycrystal [9]. An evolution equation for the fourth-order coefficient has been suggested by Böhlke [3] (see also [7,10]). In the present paper the asymptotic solutions, which are implied by the evolution equation, are derived analytically for the special case of axisymmetric

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deformations. The predictions of the phenomenological model are compared with the predictions of the Taylor polycrystal model.

The outline of the present paper is as follows. In Section 2 the effective elasticity tensor of aggregates of cubic crystallites is decomposed into a texture-independent isotropic part and a texture-dependent anisotropic part. The evolution equation for the anisotropic part is formulated based on two assumptions: first, the direction of the driving term of the evolution equation depends only on the direction of the inelastic rate of deformation, and second, the saturation is proportional to the current state of anisotropy. An advantage of the present approach is that asymptotic values for the elasticity tensors can be analytically determined for axisymmetric deformations as shown in the present paper. Furthermore, it can be shown that axial effects observed in torsion experiments can be described by the model presented here [3,10]. In Section 3 the transition from an elastically isotropic initial state to a path-dependent final anisotropic state is discussed for the case of polycrystalline copper and axisymmetric deformation paths. The asymptotic elastic properties, as they are predicted by the suggested material model, are compared with Taylor type texture simulations.

Notation. Throughout the text a direct tensor notation is preferred. In order to avoid additional formal definitions, the index notation is applied in some cases using the summation convention. A linear mapping of a second-order tensor is written as $\mathbf{A} = \mathbb{C}[\mathbf{B}]$. The scalar product, the dyadic product, and the Frobenius norm are denoted by $\mathbf{A}\cdot\mathbf{B} = \mathrm{tr}(\mathbf{A}^{\mathrm{T}}\mathbf{B})$, $\mathbf{A}\otimes\mathbf{B}$, and $\|\mathbf{A}\| = (\mathbf{A}\cdot\mathbf{A})^{1/2}$, respectively. Completely symmetric and traceless tensors of any order are designated by a prime, e.g., $\mathbf{A}'$ and $\mathbb{C}'$. $\mathbf{I}$ and $\mathbb{I}$ denote the second-order identity and the identity on symmetric second-order tensors, respectively. The Rayleigh product of a second-order tensor $\mathbf{A} = A_{ij}\mathbf{e}_i\otimes\mathbf{e}_j$ and a fourth-order tensor $\mathbb{C} = C_{ijkl}\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k\otimes\mathbf{e}_l$ is defined by

$$\mathbf{A}\star\mathbb{C} = C_{ijkl}(\mathbf{A}\mathbf{e}_i)\otimes(\mathbf{A}\mathbf{e}_j)\otimes(\mathbf{A}\mathbf{e}_k)\otimes(\mathbf{A}\mathbf{e}_l), \qquad (1)$$

where $\{\mathbf{e}_i\}$ is an arbitrary but fixed orthonormal basis. A tilde indicates that a quantity is formulated with respect to the undistorted configuration.

2. Description of the model

Hooke's Law. The stress–strain relation applied here is equivalent to the standard formulation of the theory of multiplicative elasto-plasticity based on a linear hyperelastic law of the St. Venant–Kirchhoff type. The Kirchhoff stress tensor $\boldsymbol{\tau}$ is given as a linear map of the elastic Almansi strain tensor (see, e.g., [7])

$$\boldsymbol{\tau} = \mathbb{C}_{\mathrm{e}}[\mathbf{E}^A_{\mathrm{e}}], \qquad \mathbf{E}^A_{\mathrm{e}} = \tfrac{1}{2}(\mathbf{I} - \mathbf{F}_{\mathrm{e}}^{-\mathrm{T}}\mathbf{F}_{\mathrm{e}}^{-1}), \qquad \mathbf{F}_{\mathrm{e}} = \mathbf{F}\mathbf{F}_{\mathrm{p}}^{-1}. \qquad (2)$$

The Kirchhoff stress $\boldsymbol{\tau}$ is defined by the Cauchy stress $\boldsymbol{\sigma}$ and the determinant J of $\mathbf{F}$: $\boldsymbol{\tau} = J\boldsymbol{\sigma}$. The tensor $\mathbf{F}_{\mathrm{e}}$ represents the elastic portion of the deformation gradient $\mathbf{F} = \mathbf{F}_{\mathrm{e}}\mathbf{F}_{\mathrm{p}}$. The multiplicative decomposition of the deformation gradient into an elastic and an inelastic portion can be consistently derived by the concept of isomorphic elastic ranges Bertram [1]. The stiffness operator $\mathbb{C}_{\mathrm{e}}$ is given by the Rayleigh product of $\mathbf{F}_{\mathrm{e}}$ and the reference stiffness tensor $\tilde{\mathbb{C}}$ with respect to the undistorted configuration

$$\begin{aligned}\mathbb{C}_{\mathrm{e}} &= \mathbf{F}_{\mathrm{e}}\star\tilde{\mathbb{C}} \\ &= \tilde{C}_{ijkl}(\mathbf{F}_{\mathrm{e}}\mathbf{e}_i)\otimes(\mathbf{F}_{\mathrm{e}}\mathbf{e}_j)\otimes(\mathbf{F}_{\mathrm{e}}\mathbf{e}_k)\otimes(\mathbf{F}_{\mathrm{e}}\mathbf{e}_l). \qquad (3)\end{aligned}$$

Within the standard formulation of the theory of elasto-plasticity, i.e. without deformation induced elastic anisotropy, $\tilde{\mathbb{C}}$ is assumed to be constant. Eq. (3) indicates that the tensor $\mathbb{C}_{\mathrm{e}}$ is generally non-constant, even in the case if $\tilde{\mathbb{C}}$ is constant and isotropic. In the present paper $\tilde{\mathbb{C}}$ depends on the crystallographic texture in the aggregate, which evolves for large inelastic deformations. The rate of change of $\tilde{\mathbb{C}}$ is specified by an evolution equation.

Decomposition of Hooke's tensor. Volume or orientational averages of stiffness tensors with cubic symmetry generally allow for the following unique decomposition of the effective properties into three different parts

$$\begin{aligned}&\tilde{\mathbb{C}} = 3K\mathbb{P}_1^I + 2G\mathbb{P}_2^I + \zeta\tilde{\mathbb{A}}', \\ &\mathbb{P}_1^I = \tfrac{1}{3}\mathbf{I}\otimes\mathbf{I}, \qquad \mathbb{P}_2^I = \mathbb{I} - \mathbb{P}_1^I. \qquad (4)\end{aligned}$$

This result holds, if the arithmetic, the harmonic mean, or the geometric mean of stiffness tensors is determined [5,6]. The two fourth-order projectors $\mathbb{P}_1^I$ and $\mathbb{P}_2^I$ represent the isotropic part of the elastic law, which is independent of the crystallographic texture in the aggregate. K is the bulk modulus and G is the shear modulus.

The tensor $\widetilde{\mathbb{A}}'$ represents the purely anisotropic part of the stiffness tensor $\widetilde{\mathbb{C}}$. If there is no crystallographic texture in the aggregate, then $\widetilde{\mathbb{A}}' = \mathbb{O}$ holds and $\widetilde{\mathbb{C}}$ is isotropic. For the above mentioned orientation averages and aggregates of cubic crystals, the anisotropic part $\widetilde{\mathbb{A}}'$ is independent of the special type of averaging and is given by

$$\widetilde{\mathbb{A}}' = \frac{\sqrt{30}}{30}\left(\mathbf{I}\otimes\mathbf{I} + 2\mathbb{I} - \int_g f(g)\sum_{i=1}^{3}\tilde{\mathbf{g}}_i(g) \otimes \tilde{\mathbf{g}}_i(g)\otimes\tilde{\mathbf{g}}_i(g)\otimes\tilde{\mathbf{g}}_i(g)\,\mathrm{d}g\right), \tag{5}$$

where $\{\tilde{\mathbf{g}}_i\}$ denotes the lattice vectors of the single crystals on the microscale [5,6]. $f(g)$ represents the crystal orientation distribution function. The function f specifies the volume fraction of crystals having an orientation g, i.e. $\mathrm{d}V(g)/V = f(g)\,\mathrm{d}g$. From (5) one concludes that the Frobenius norm of $\widetilde{\mathbb{A}}'$ is equal to one for a single crystal orientation. Furthermore, from (5) one deduces that the tensor $\widetilde{\mathbb{A}}'$ is irreducible [12,14], i.e. symmetric and traceless with respect to all pairs of indices

$$\widetilde{A}'_{ijkl} = \widetilde{A}'_{jikl} = \widetilde{A}'_{klij} = \widetilde{A}'_{kjil} = \cdots, \quad \widetilde{A}'_{iikl} = 0 \tag{6}$$

Ref. [7]. Because of the aforementioned constraints, $\widetilde{\mathbb{A}}'$ contains only nine independent components, and this property leads to a rigorous simplification of the evolution equation for $\widetilde{\mathbb{C}}'$. The reduction of the number of independent elastic constants on the macroscale from 21 (general triclinic) to 11 (reduced triclinic) is caused by the cubic crystal symmetry on the microscale. The equation $\widetilde{\mathbb{A}}' = \mathbb{O}$ represents the isotropy condition for elasticity in terms of crystal orientations. This equation can be solved exactly for all even integers $N \geqslant 4$ [2,8]. The tensor $\widetilde{\mathbb{A}}'$ represents the fourth-order coefficient of a tensorial Fourier expansion of the crystal orientation distribution function of an aggregate of cubic crystals [9]. In the subsequent considerations the components of fourth-order tensors refer to the orthonormal basis $\mathbf{B}_\alpha$ of symmetric second-order tensors [13].

$$\begin{aligned}
\mathbf{B}_1 &= \mathbf{e}_1\otimes\mathbf{e}_1, & \mathbf{B}_4 &= \frac{\sqrt{2}}{2}(\mathbf{e}_2\otimes\mathbf{e}_3 + \mathbf{e}_3\otimes\mathbf{e}_2),\\
\mathbf{B}_2 &= \mathbf{e}_2\otimes\mathbf{e}_2, & \mathbf{B}_5 &= \frac{\sqrt{2}}{2}(\mathbf{e}_1\otimes\mathbf{e}_3 + \mathbf{e}_3\otimes\mathbf{e}_1),\\
\mathbf{B}_3 &= \mathbf{e}_3\otimes\mathbf{e}_3, & \mathbf{B}_6 &= \frac{\sqrt{2}}{2}(\mathbf{e}_1\otimes\mathbf{e}_2 + \mathbf{e}_2\otimes\mathbf{e}_1).
\end{aligned} \tag{7}$$

Evolution equation of $\widetilde{\mathbb{A}}'$. For the evolution of $\widetilde{\mathbb{A}}'$ the following ansatz is applied

$$\dot{\widetilde{\mathbb{A}}}' = \|\widetilde{\mathbf{D}}'_{\mathrm{p}}\|(\widetilde{\mathbb{G}}'(\widetilde{\mathbf{N}}'_{\mathrm{p}}) - d(III_{\mathrm{p}})\widetilde{\mathbb{A}}'), \quad III_{\mathrm{p}} = \det(\widetilde{\mathbf{N}}'_{\mathrm{p}}). \tag{8}$$

The driving term $\widetilde{\mathbb{G}}'(\widetilde{\mathbf{N}}'_{\mathrm{p}})$ depends only on the direction $\widetilde{\mathbf{N}}'_{\mathrm{p}}$ of the inelastic rate of deformation $\widetilde{\mathbf{D}}'_{\mathrm{p}}$, which is the symmetric part of $\dot{\mathbf{F}}_{\mathrm{p}}\mathbf{F}_{\mathrm{p}}^{-1}$. The ansatz (8) takes into account that for constant strain rates the crystallographic texture tends to saturate for large deformations. The saturation behavior is controlled by the term $d(III_{\mathrm{p}})\widetilde{\mathbb{A}}'$. The function $d(III_{\mathrm{p}})$ depends on the only non-constant principal invariant of $\widetilde{\mathbf{N}}'_{\mathrm{p}}$, i.e., its determinant.

The general form of the function $\widetilde{\mathbb{G}}'$ can be determined by the theory of isotropic tensor functions [7]. Due to the constraint that, similar to $\widetilde{\mathbb{A}}'$, the function $\widetilde{\mathbb{G}}'$ has to be symmetric and traceless with respect to all pairs of indices, the function $\widetilde{\mathbb{G}}'$ is given by three fourth-order tensor generators $\widetilde{\mathbb{G}}'_\alpha$, and corresponding scalar functions G_α

$$\widetilde{\mathbb{G}}'(\widetilde{\mathbf{N}}'_{\mathrm{p}}) = G_1(III_{\mathrm{p}})\widetilde{\mathbb{G}}'_1(\widetilde{\mathbf{N}}'_{\mathrm{p}}) + G_2(III_{\mathrm{p}})\widetilde{\mathbb{G}}'_2(\widetilde{\mathbf{N}}'_{\mathrm{p}}) + G_3(III_{\mathrm{p}})\widetilde{\mathbb{G}}'_3(\widetilde{\mathbf{N}}'_{\mathrm{p}}). \tag{9}$$

The tensor generators are given by

$$\mathbb{G}'_1(\mathbf{A}) = \langle\mathbf{A}\otimes\mathbf{A}\rangle - \tfrac{2}{7}(\mathrm{tr}(\mathbf{A})\langle\mathbf{A}\otimes\mathbf{I}\rangle + 2\langle\mathbf{A}^2\otimes\mathbf{I}\rangle) + \tfrac{1}{35}(\mathrm{tr}(\mathbf{A})^2 + 2\mathrm{tr}(\mathbf{A}^2))\langle\mathbf{I}\otimes\mathbf{I}\rangle, \tag{10}$$

$$\mathbb{G}'_2(\mathbf{A}) = \mathbb{G}'_1(\mathbf{A}^2), \tag{11}$$

$$\mathbb{G}'_3(\mathbf{A}) = \langle\mathbf{A}^2\otimes\mathbf{A}\rangle - \tfrac{2}{7}(\mathrm{tr}(\mathbf{A}^2)\langle\mathbf{A}\otimes\mathbf{I}\rangle + \mathrm{tr}(\mathbf{A})\langle\mathbf{A}^2\otimes\mathbf{I}\rangle + 4\langle\mathbf{A}^3\otimes\mathbf{I}\rangle) + \tfrac{2}{35}(\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{A}^2) + 2\mathrm{tr}(\mathbf{A}^3))\langle\mathbf{I}\otimes\mathbf{I}\rangle. \tag{12}$$

The bracket formula $\langle\cdot\rangle$ is defined by $(\mathbf{A},\mathbf{B} \in \mathrm{Sym})$

$$\langle A_{ij}B_{kl}\rangle = \tfrac{1}{6}(A_{ij}B_{kl} + A_{ik}B_{jl} + A_{il}B_{kj} + B_{ij}A_{kl} + B_{ik}A_{jl} + B_{il}A_{kj}). \tag{13}$$

Asymptotic solutions for $\widetilde{\mathbb{A}}'$. The stretching tensor, which is assumed to be traceless, can generally be decomposed as

$$\mathbf{D} = \|\mathbf{D}\|\mathbf{Q}_{\mathrm{D}}\mathbf{N}_{\mathrm{D}}\mathbf{Q}_{\mathrm{D}}^{\mathrm{T}}, \tag{14}$$

where $\|\mathbf{D}\|$ is the magnitude of $\mathbf{D}$, and $\mathbf{Q}_{\mathrm{D}}$ is an orthogonal tensor that rotates an arbitrary but fixed orthonormal reference basis $\{\mathbf{e}_i\}$ onto the eigenvectors of $\mathbf{D}$. The constraints, i.e., that $\mathbf{N}_{\mathrm{D}}$ is traceless and has a magnitude equal to one, are identically fulfilled by the following parameterization of $\mathbf{N}_{\mathrm{D}}$

$$\mathbf{N}_{\mathrm{D}} = \sum_{\alpha=1}^{3} n_\alpha \mathbf{e}_\alpha \otimes \mathbf{e}_\alpha, \qquad n_{1,3} = -\frac{\sqrt{6}}{6}\xi \pm \frac{\sqrt{2}}{2}\sqrt{1-\xi^2}, \qquad n_2 = \frac{\sqrt{6}}{3}\xi, \tag{15}$$

$\xi \in [-1/2, +1/2]$. The ξ-values −0.5, 0, and +0.5 belong to the uniaxial tension, plane strain compression, and simple compression, respectively.

Based on Eq. (8), the asymptotic values $\widetilde{\mathbb{A}}'_\infty$ of $\widetilde{\mathbb{A}}'$ for a constant plastic flow can be determined. The elastic strains are small in the case of metal elasticity. Furthermore, texture simulations indicate that uniaxial macroscopic deformations do not induce rotations of the symmetry axes of the effective elasticity tensor [3]. Hence, $\widetilde{\mathbf{F}} \approx \mathbf{I}$, $\mathbf{F} \approx \mathbf{F}_{\mathrm{p}}$, and $\widetilde{\mathbf{D}}_{\mathrm{p}} \approx \mathbf{D}$ hold. Because of the last approximation, a flow rule has not been considered in this paper. For a flow rule, which includes the anisotropy in terms of $\widetilde{\mathbb{A}}'$, see, e.g., [10].

With the aforementioned assumptions Eq. (8) is

$$\dot{\widetilde{\mathbb{A}}}' = \|\mathbf{D}'\|\left(\sum_{i=1}^{3} G_i(III)\mathbb{G}'_i(\mathbf{N}') - d(III)\widetilde{\mathbb{A}}'\right), \tag{16}$$

where $III = \det(\mathbf{N}')$ and $\mathbf{N}' = \mathbf{D}'/\|\mathbf{D}'\|$. If $\dot{\widetilde{\mathbb{A}}}'$ vanishes we have

$$\widetilde{\mathbb{A}}'_\infty = \frac{1}{d(III)}\sum_{i=1}^{3} G_i(III)\mathbb{G}'_i(\mathbf{N}'). \tag{17}$$

Inspection of the tensors $\mathbb{G}'_i$ shows that for the special values $\xi = \pm 0.5$, i.e. for simple tension and simple compression, the tensors $\mathbb{G}'_1$, $\mathbb{G}'_2$, and $\mathbb{G}'_3$ are linearly dependent. Hence for these axisymmetric deformation modes the directions of the asymptotic values $\widetilde{\mathbb{A}}'_\infty$ are independent of the scalar functions $G_{1,2,3}$. For the tension mode ($\xi = -0.5$) one obtains with (15) and (16)

$$\widetilde{\mathbb{A}}'_\infty = \frac{k}{840}\begin{bmatrix} +8.0 & -4.0 & -4.0 & 0 & 0 & 0 \\ \cdot & +3.0 & +1.0 & 0 & 0 & 0 \\ \cdot & \cdot & +3.0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & +2.0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & -8.0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & -8.0 \end{bmatrix} \times \mathbf{B}_\alpha \otimes \mathbf{B}_\beta \tag{18}$$

with $k = \sqrt{6}(6\sqrt{6}G_1(III) + \sqrt{6}G_2(III) + 12G_3(III))/d(III)$ and $III = \sqrt{16}/8$. For the compression mode ($\xi = +0.5$) one obtains with (15) and (16)

$$\widetilde{\mathbb{A}}'_\infty = \frac{k}{840}\begin{bmatrix} +3.0 & +1.0 & -4.0 & 0 & 0 & 0 \\ \cdot & +3.0 & -4.0 & 0 & 0 & 0 \\ \cdot & \cdot & +8.0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & -8.0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & -8.0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & +2.0 \end{bmatrix} \times \mathbf{B}_\alpha \otimes \mathbf{B}_\beta \tag{19}$$

with $k = \sqrt{6}(6\sqrt{6}G_1(III) + \sqrt{6}G_2(III) - 12G_3(III))/d(III)$ and $III = -\sqrt{16}/8$. The $G_{1,2,3}$ and d should satisfy the condition $k \neq 0$ and $k \neq \infty$. One can conclude that the growth law (8) allows for an explicit determination of the asymptotic values of the elastic anisotropy in the case of axisymmetric deformations. The asymptotic values (18) and (19) have a transverse isotropic symmetry.

3. Numerical results

The Taylor–Lin model. For an investigation of the monotonic stress–strain response and the respective texture development, usually four types of experiments are used: uniaxial extension, channel-die compression, uniaxial compression, and simple shear [16]. [11] evaluated the Taylor model for the aforementioned types of deformation by comparing the predictions of the stress–strain curves and the evolution of crystallographic textures in initially isotropic OFHC copper. The agreement between experimental and simulated pole figures is shown to be good or reasonable.

For the Taylor type texture simulations a linear elastic law with cubic symmetry and a flow rule with 12 octahedral slip systems is applied. In the slip systems an overstress model is used. The material equations and parameters are given in [3]. The texture simulations discussed in subsequent paragraphs have been performed with 1000 filtered random orientations [4]. For the subsequent considerations the amount of deformation is quantified by the v.Mises equivalent strain $\phi(t) = \sqrt{2/3}\int_0^t \|\mathbf{D}(\tilde{t})\|\,d\tilde{t}$.

Uniaxial tension. The direction of the velocity gradient (denoted by $\|$) is given by $\mathbf{L}(\xi = -0.5)\| 0.8\mathbf{e}_1 \otimes \mathbf{e}_1 - 0.4(\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3)$. For initially non-textured aggregates, such a deformation induces an axisymmetric texture. In agreement with the experiments, the Taylor type simulations indicate that in uniaxial tension there are two dominant texture components. Either the $\{1\,1\,1\}$ planes or the $\{1\,0\,0\}$ planes become perpendicular to the loading axis. In the case of uniaxial tension, the components of $\mathbb{A}_e' = \widetilde{\mathbf{F}}\star\widetilde{\mathbb{A}}'$ are approximately constant for equivalent strain values ϕ larger than 2. The maximum amount of anisotropy is given by $\|\mathbb{A}_e'\| \approx 0.2$. The effective Eulerian stiffness tensor for an equivalent strain of $\phi = 2.5$, as it is predicted by the Taylor–Lin model, is given by

$$\mathbb{C}^{\text{Taylor}}_{e(\xi=-0.5,\phi=2.5)} = \begin{bmatrix} 220.0 & 95.37 & 95.38 & 0.01 & -0.05 & 0.10 \\ \cdot & 213.6 & 101.9 & 0.17 & -0.06 & 0.25 \\ \cdot & \cdot & 213.6 & 0.15 & 0.11 & -0.35 \\ \cdot & \cdot & \cdot & 111.7 & -0.50 & -0.09 \\ \cdot & \cdot & \cdot & \cdot & 98.76 & 0.01 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 98.74 \end{bmatrix} \times \text{GPa}\mathbf{B}_\alpha \otimes \mathbf{B}_\beta, \tag{20}$$

If the anisotropic portion of $\mathbb{C}^{\text{Taylor}}_{e(\xi=-0.5,\phi=2.5)}$ is normalized by a factor α such that it has the same magnitude as $840\widetilde{\mathbb{A}}'_\infty/k$ in Eq. (18) then one obtains

$$\alpha\mathbb{A}'^{\text{Taylor}}_{e(\xi=-0.5,\phi=2.5)} = \begin{bmatrix} 7.9 & -4.0 & -4.0 & 0.01 & -0.04 & 0.08 \\ \cdot & 3.0 & 1.0 & -0.13 & -0.05 & 0.19 \\ \cdot & \cdot & 3.0 & 0.12 & 0.09 & -0.28 \\ \cdot & \cdot & \cdot & 2.0 & -0.39 & -0.07 \\ \cdot & \cdot & \cdot & \cdot & -8.0 & 0.01 \\ \cdot & \cdot & \cdot & \cdot & \cdot & -8.0 \end{bmatrix} \times \mathbf{B}_\alpha \otimes \mathbf{B}_\beta. \tag{21}$$

Note that the components of the stiffness tensors refer to the orthonormal basis $\mathbf{B}_\alpha$. With respect to this basis, e.g., the C_{44} component is equal to $2C_{2323}$, which differs from Voigt's original notation by the factor of 2. As shown above for $\xi = -0.5$ the direction of the asymptotic value $\widetilde{\mathbb{A}}'_\infty$ depends on the functions G_α only through a scalar factor (see Eq. (18)). This implication of the phenomenological model is verified in the context of a Taylor type approach.

Uniaxial compression ($\xi = +0.50$). The direction of the velocity gradient is given by $\mathbf{L}(\xi = 0.5)\|0.4(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2) - 0.8\mathbf{e}_3 \otimes \mathbf{e}_3$. Experimental textures indicate that in the simple bulk mode there is one dominant component of texture. The grains of the polycrystal are aligned such that $\{1\,1\,0\}$ planes become perpendicular to the loading axis. The Taylor–Lin model reproduces these findings. The texture is axisymmetric with respect to the loading axis. The elastic anisotropy saturates for equivalent strains of $\phi \approx 5$. The amount of anisotropy is then given by $\|\mathbb{A}_e'\| \approx 0.185$. The effective Eulerian stiffness tensor according to the Taylor type model is

$$\mathbb{C}^{\text{Taylor}}_{e(\xi=0.5,\phi=5)} = \begin{bmatrix} 212.8 & 101.1 & 96.9 & -0.04 & 0.0 & 0.19 \\ \cdot & 212.5 & 97.2 & -0.39 & 0.06 & -0.11 \\ \cdot & \cdot & 216.8 & 0.44 & -0.07 & -0.07 \\ \cdot & \cdot & \cdot & 102.3 & -0.10 & 0.09 \\ \cdot & \cdot & \cdot & \cdot & 101.7 & -0.06 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 110.2 \end{bmatrix} \times \text{GPa}\mathbf{B}_\alpha \otimes \mathbf{B}_\beta. \tag{22}$$

If the anisotropic portion of $\mathbb{C}^{\text{Taylor}}_{e(\xi=0.5,\phi=5)}$ is normalized by a factor α such that it has the same magnitude as $840\widetilde{\mathbb{A}}'_{\infty}/k$ in Eq. (19) then one obtains

$$\alpha\widetilde{\mathbb{A}}'^{\text{Taylor}}_{e(\xi=0.5,\phi=5)} = \begin{bmatrix} 3.50 & 0.60 & -4.14 & -0.04 & 0.0 & 0.21 \\ \cdot & 3.16 & -3.80 & -0.44 & 0.07 & -0.12 \\ \cdot & \cdot & 7.98 & 0.5 & -0.08 & -0.08 \\ \cdot & \cdot & \cdot & -7.65 & -0.11 & 0.10 \\ \cdot & \cdot & \cdot & \cdot & -8.32 & -0.07 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1.21 \end{bmatrix} \times \mathbf{B}_\alpha \otimes \mathbf{B}_\beta. \qquad (23)$$

In contrast to the case $\xi = -0.5$ the Taylor–Lin model reaches the theoretical value at a large equivalent strain. The agreement of the predictions of the phenomenological model and the Taylor model are less pronounced compared to the case of uniaxial tension. This can be explained as follows. The textures predicted by the Taylor polycrystal model differ for both deformation modes. The phenomenological model predicts identical tensorial directions for the tensor $\widetilde{\mathbb{A}}'$ for uniaxial tension and compression. It can be concluded that the phenomenological model needs some improvement in order to describe the effective elasticity for the case of uniaxial compression as accurately as in case of uniaxial tension.

As already mentioned, the tensor $\widetilde{\mathbb{A}}'$ represents the fourth-order coefficient of a tensorial Fourier expansion of the crystal orientation distribution function of an aggregate of cubic crystals [9]. A material model, which aims to describe the anisotropy caused by a crystallographic texture, should at least be able to model the fourth-order coefficient, which governs the effective elastic properties. The comparison of the predictions of the Taylor model for axisymmetric deformations with the analytical results derived from the phenomenological evolution Eq. (8) shows a good or reasonable agreement. Therefore, the ansatz for the evolution equation of the fourth-order coefficient $\widetilde{\mathbb{A}}'$ of the crystal orientation distribution function is supported by the Taylor simulations.

4. Summary

The asymptotic values of elastic anisotropy of polycrystalline copper have been analyzed for axisymmetric deformations processes. A phenomenological model for the evolution of the texture-dependent effective elastic properties has been used to predict the corresponding saturation values. The predictions are in accordance with the values given by a Taylor type polycrystal simulation.

References

[1] A. Bertram, Description of finite inelastic deformations, in: In: A. Benallal, R. Billardon, D. Marquis (Eds.), Proceedings of MECAMAT'92, International Seminar on Multiaxial Plasticity, 1992.

[2] A. Bertram, T. Böhlke, N. Gaffke, B. Heiligers, R. Offinger, On the generation of discrete isotropic orientation distributions for linear elastic cubic crystals, J. Elast. 58 (2001) 233–248.

[3] T. Böhlke, Crystallographic Texture Evolution and Elastic Anisotropy: Simulation, Modeling, and Applications, Shaker Verlag, Dissertation, Fakultät für Maschinenbau, Otto-von-Guericke-Universität Magdeburg, 2001.

[4] T. Böhlke, A. Bertram, Simulation of texture development and induced anisotropy of polycrystals, in: S. Atluri, P. O'Donoghue (Eds.), Proceedings of ICES'98, Modelling and Simulation Based Engineering, 1998.

[5] T. Böhlke, A. Bertram, A minimum problem defining effective isotropic elastic properties, Z. Angew. Math. Mech. 80 (S2) (2000) S419–S420.

[6] T. Böhlke, A. Bertram, Bounds for the geometric mean of 4th-order elasticity tensors with cubic symmetry, Z. Angew. Math. Mech. 81 (S2) (2001) S333–S334.

[7] T. Böhlke, A. Bertram, The evolution of Hooke's law due to texture development in polycrystals, Int. J. Solids Struct. 38 (52) (2001) 9437–9459.

[8] T. Böhlke, A. Bertram, On isotropic orientation distributions of cubic crystals, J. Mech. Phys. Solids 49 (11) (2001) 2459–2470.

[9] T. Böhlke, A. Bertram, Crystallographic texture induced anisotropy in copper: An approach based on a tensorial Fourier expansion of the codf, Proceedings of the Sixth European Mechanics of Materials Conference (EMMC6), 2002, pp. 273–280.

[10] T. Böhlke, A. Bertram, E. Krempl, Modeling nonlinear effects in free-end torsion. Int. J. Plast. (2003), in press.

[11] C. Bronkhorst, S. Kalidindi, L. Anand, Polycrystalline plasticity and the evolution of crystallographic texture in fcc metals, R. Soc. Lond. A 341 (1992) 443–477.

[12] H. Ehrentraut, W. Muschik, On symmetric irreducible tensors in d-dimensions, ARI 51 (1998) 149–159.

[13] F. Federov, Theory of Elastic Waves in Crystals, Plenum Press, New York, 1968.
[14] S. Forte, M. Vianello, Symmetry classes for elasticity tensors, J. Elast. 43 (1996) 81–108.
[15] M. Guidi, B.L. Adams, E.T. Onat, Tensorial representation of the orientation distribution function in cubic polycrystals, Textures Microstruct. 19 (1992) 147–167.
[16] U. Kocks, C. Tome, H. Wenk, Texture and Anisotropy: Preferred Orientations in Polycrystals and Their Effect on Materials Properties, Cambridge University Press, 1998.
[18] C. Stickels, P. Mould, The use of Young's modulus for predicting the plastic-strain ratio of low-carbon steel sheets, Met. Trans. 1 (1970) 1303–1312.
[19] J. Weerts, Elastizität von Kupferblechen, Z. Metallk. 5 (1933) 101–103.