# Symmetry properties of fourth-order tensors with applications in continuum mechanics

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**SUMMARY**

**In many physical and technical applications linear mappings between symmetric tensors are needed. Their properties and representations are shown in order to imbed them in the set of fourth order tensors, avoiding desadvantages of the usual methods.**

## 1. INTRODUCTION

In the mechanical theory of continuous bodies linear relations between symmetric tensors play an important role. In the calculus of tensors the representation of a linear relation between second-order tensors is a tetrad or a fourth-order tensor. Formally we write

$$\mathbf{A} = \mathbf{C}^{a}\,\mathbf{B}$$

where

**B** is the argument, a second order symmetric tensor
**A** is the value, also a symmetric tensor, and
**C** is a tetrad, that relates **A** and **B** in a linear manner.

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Examples of such relations are:

1) In the linear theory of elasticity

- **A** the stress tensor
- **B** the infinitesimal deformation tensor
- **C** the elasticity tetrad

2) In the incremental theory of finite elasticity

- **A** the increment of an appropiate stress tensor
- **B** the increment of deformation
- **C** the instantaneous elasticity tensor

3) In the linear theory of viscosity

- **A** the stress tensor
- **B** the rate of deformation tensor
- **C** the viscosity tensor

In all of these examples **A** and **B** are symmetric second-order tensors for physical reasons.

In order to assure the symmetry of the value under **C** it is customary to impose certain symmetry properties on **C**. As a consequence of this symmetry **C** turns out to be singular, which surely is an important disadvantage of this procedure, although hardly mentioned in the literature.

In the present paper we will clarify this somehow unfortunate consequence and suggest different possibilities to remove or avoid it.

## 2. BASIC NOTATIONS

In this paper we will use an index-free notation as well as a component notation referred to a cartesian vector-base $< \mathbf{e}_i >$ of the underlying three-dimensional vector-space.

With summation over repeated indices we obtain the following representations:

- for vectors $\mathbf{u} = u_i \, \mathbf{e}_i \, , \; \mathbf{v} = v_i \, \mathbf{e}_i \, , \dots$
- for tensors $\mathbf{A} = a_{ij} \, \mathbf{e}_i \circ \mathbf{e}_j \, , \; \mathbf{B} = b_{ij} \, \mathbf{e}_i \circ \mathbf{e}_j \, , \dots$

with the tensor product «o». The inner product between

vectors is $\mathbf{u} \cdot \mathbf{v} = u_i \, v_i$

and the left and right-hand applications of tensors to vectors are denoted by

$$\mathbf{A} \,.\, \mathbf{u} = a_{ij}\, u_j\, \mathbf{e}_i$$

$$\mathbf{u} \,.\, \mathbf{A} = u_i\, a_{ij}\, \mathbf{e}_j \;,$$

respectively. The combination of two tensors is

$$\mathbf{A} \,.\, \mathbf{B} = a_{ij}\, b_{jk}\, \mathbf{e}_i \circ \mathbf{e}_k$$

and the inner product between two tensors is defined by the trace

$$\mathbf{A} \bullet \mathbf{B} = \mathrm{tr}\,(\mathbf{A} \,.\, \mathbf{B}^t) = a_{ij}\, b_{ij} \;,$$

Where $\mathbf{B}^t$ is the transpose of **B**. A tetrad or a fourth-order tensor is

$$\mathbf{C} = c_{ijkl}\, \mathbf{e}_i \circ \mathbf{e}_j \circ \mathbf{e}_k \circ \mathbf{e}_l$$

with its left and right-hand applications to second-order tensors

$$\mathbf{C} \bullet \mathbf{A} = c_{ijkl}\, a_{kl}\, \mathbf{e}_i \circ \mathbf{e}_j$$

$$\mathbf{A} \bullet \mathbf{C} = a_{ij}\, c_{ijkl}\, \mathbf{e}_k \circ \mathbf{e}_l \;.$$

The combination of tetrads is defined by

$$\mathbf{C} \bullet \mathbf{D} = c_{ijkl}\, d_{klmn}\, \mathbf{e}_i \circ \mathbf{e}_j \circ \mathbf{e}_m \circ \mathbf{e}_n \;,$$

and the linear operations on each level are

$$\mathbf{u} + \alpha\, \mathbf{v} = (u_i + \alpha v_i)\, \mathbf{e}_i$$

$$\mathbf{A} + \alpha\, \mathbf{B} = (a_{ij} + \alpha b_{ij})\, \mathbf{e}_i \circ \mathbf{e}_j$$

$$\mathbf{C} + \alpha\, \mathbf{D} = (c_{ijkl} + \alpha\, d_{ijkl})\, \mathbf{e}_i \circ \mathbf{e}_j \circ \mathbf{e}_k \circ \mathbf{e}_l$$

The inverse of a tensor and a tetrad are, respectively,

$$\mathbf{A}^{-1} \text{ and } \mathbf{C}^{-1} \,.$$

We call a tensor of any order singular, if the inverse does not exist.

The transpose of a tensor is

$$\mathbf{A}^t = a_{ji}\, \mathbf{e}_i \circ \mathbf{e}_j$$

DEL PIERO has introduced the transposition tetrad $\mathbf{S}$, such that

$$\mathbf{A}^t = \mathbf{S} \bullet \mathbf{A}$$

holds for each tensor. The component form of $\mathbf{S}$ is

$$\mathbf{S} = \delta_{il}\, \delta_{jk}\, \mathbf{e}_i \odot \mathbf{e}_j \odot \mathbf{e}_k \odot \mathbf{e}_l \,.$$

It is evident, that

$$\mathbf{A}^t = \mathbf{A} \bullet \mathbf{S}$$

and

$$\mathbf{A} = \mathbf{S} \bullet \mathbf{S} \bullet \mathbf{A} = \mathbf{A} \bullet \mathbf{S} \bullet \mathbf{S}$$

hold for each tensor A, and therefore

$$\mathbf{S} \bullet \mathbf{S} = \mathbf{I}\,,$$

the fourth-order identity, or $\mathbf{S} = \mathbf{S}^{-1}$.

## 3. SYMMETRY DEFINITIONS OF TENSORS

A (second-order) tensor **A** is said to be symmetric, if

$$\mathbf{u} \,.\, (\mathbf{A} \,.\, \mathbf{v}) = \mathbf{v} \,.\, (\mathbf{A} \,.\, \mathbf{u})$$

holds for all vectors $\mathbf{u}$ and $\mathbf{v}$. Clearly, this is equivalent to the conditions

$$\mathbf{A} = \mathbf{A}^t \quad \text{or} \quad a_{ij} = a_{ji}$$

for all subindices. A tensor is called antisymmetric, if

$$\mathbf{u}.(\mathbf{A} \,.\, \mathbf{v}) = -\, \mathbf{v}.(\mathbf{A} \,.\, \mathbf{u})$$

or

$$\mathbf{A} = -\mathbf{A}^t \quad \text{or} \quad a_{ij} = -\, a_{ji}\,.$$

Analogously, we define a tetrad to be symmetric, if

$$\mathbf{A} \bullet (\mathbf{C} \bullet \mathbf{B}) = \mathbf{B} \bullet (\mathbf{C} \bullet \mathbf{A})$$

for all tensors **A** and **B**. This is equivalent to the component form

$$c_{ijkl} = c_{klij}\,.$$

Apart from the symmetry, we need two more definitions:

The tetrad $\mathbf{C}$ is left-subsymmetric, if

$$\mathbf{A} \bullet \mathbf{C} = \mathbf{A}^t \bullet \mathbf{C}$$

and right-subsymmetric, if

$$\mathbf{C} \bullet \quad = \mathbf{C} \bullet \mathbf{A}^t$$

hold for all tensors **A**. The equivalent component forms are

$$c_{ijkl} = c_{jikl}$$

and

$$c_{ijkl} = c_{ijlk} \quad ,$$

respectively. The following propositions are inmediate consequences of these definitions.

**Proposition 1.-** A tetrad $\mathbf{C}$ is left-subsymmetric, if, and only if the tensor $\mathbf{C} \bullet \mathbf{A}$ is symmetric for all tensors **A** . $\mathbf{C}$ is right-subsymmetric, if, and only if, the tensor $\mathbf{A} \bullet \mathbf{C}$ is symmetric for all tensors **A**.

**Proof.**

$$\begin{aligned} c_{ijkl} = c_{jikl} \quad \leftrightarrow \quad \mathbf{C} \bullet \mathbf{A} &= c_{ijkl}\, a_{kl}\, \mathbf{e}_i \odot \mathbf{e}_j \\ &= c_{jikl}\, a_{kl}\, \mathbf{e}_i \circ \mathbf{e}_j \\ &= (\mathbf{C} \bullet \mathbf{A})^t \end{aligned}$$

The proof of the second part is completely analogous. □

**Proposition 2.-** A tetrad is left-subsymmetric, if and only if $\mathbf{A} \bullet \mathbf{C}$ is the zero-tensor for any antisymmetric tensor **A**, and right-subsymetric, if the same holds for $\mathbf{C} \bullet \mathbf{A}$.

**Proof.** Let **A** be antisymmetric and $\mathbf{C}$ left-sybsymmetric. Then

$$\mathbf{A} \bullet \mathbf{C} = a_{ij}\, c_{ijkl}\, \mathbf{e}_k \circ \mathbf{e}_l = -\, a_{ji}\, c_{ijkl}\, \mathbf{e}_k \circ \mathbf{e}_l$$

$$= -\, a_{ji}\, c_{jikl}\, \mathbf{e}_k \circ \mathbf{e}_l = -\, \mathbf{A} \bullet \mathbf{C}$$

and hence, is zero. On the other hand, for any tensor **T**, the tensor $\mathbf{T} - \mathbf{T}^t$ is antisymmetric. If

$$(\mathbf{T} - \mathbf{T}^t) \bullet \mathbf{C} = 0$$

results $\mathbf{T} \bullet \mathbf{C} = \mathbf{T}' \bullet \mathbf{C}$, and thus **C** is left subsymmetric. The proof of the second part is completely analogous. □

**Proposition 3.-** If a tetrad is left or right-subsymmetric, it is singular.

**Proof.** If **C** has one of the subsymmetries, it maps all antisymmetric tensors into **0**. Clearly, a mapping is singular, if its value is zero for more than one argument. □

**Proposition 4.-** For a symmetric tetrad the left and right subsymmetry coincide.

**Proof.** Let us assume, that **C** is symmetric and left subsymmetric. Then

$$c_{ijkl} = c_{jikl} = c_{klij} = c_{klji}$$

and hence, **C** is also right-subsymmetric, and vice versa. □

In the literature (e.g. SOKOLNIKOFF) it is customary to claim symmetry properties for **C** in the folowing manner:

— In order to obtain symmetric values for

$$\mathbf{A} = \mathbf{C} \bullet \mathbf{B}$$

**C** is supposed to be left-subsymmetric. In the given examples **A** is symmetric due to the balance of angular momentum.

— If **B** is symmetric, one can impose the right-subsymmetry on **C** without affecting the physically relevant part of **C**. This can be considered as a normalization, because by experiment one cannot determine the value of non-symmetric tensors **B**.

— In many cases **C** is moreover symmetric. In elasticity this is, e.g., due to the existence of a potential energy. Clearly, in combination with the left-subsymmetry, **C** automatically satisfies the right-subsymmetry in this case.

However, imposing either subsymmetry on the tetrade **C**, this turns out to be singular, a fact which is not acceptable, as one frequently needs the inverse of **C** which plays the role of the flexibility tensor in elasticity, for example. In order to avoid this unpleasant situation, we will suggest two alternative procedures, the latter of which is fue to VOIGT.

## 4. FIRST SUGGESTION

Considering symmetric arguments and symmetric values as the physically 'real' part of the linear relations, and the antisymmetric parts as the 'unreal' ones, we want to avoid interferences between the 'real' and the 'unreal' parts caused by the tetrad.

Hence, we postulate that

— 'real' arguments have only 'real' values, and
— 'unreal' arguments have only 'unreal' values.

This is achieved under the following conditions:

**Definition:**

A tetrad is called to obey the condition
— (S1), if it maps symmetric tensors into symmetric tensors,
— (S2), if it maps antisymmetric tensors into antisymmetric tensors.

**Theorem 1:**

For any tetrad $\mathbf{C}$, the condition (S1) is equivalent to the condition

$$\mathbf{C} + \mathbf{C} \bullet \mathbf{S} = \mathbf{S} \bullet \mathbf{C} + \mathbf{S} \bullet \mathbf{C} \bullet \mathbf{S}$$

which has the component form

$$c_{ijkl} + c_{ijlk} = c_{jikl} + c_{jilk} \,.$$

**Proof.** For any tetrad $\mathbf{C}$, the condition (S1) is equivalent to

$$\mathbf{C} \bullet (\mathbf{T} + \mathbf{T}') = [\mathbf{C} \bullet (\mathbf{T} + \mathbf{T}')]' \quad \text{for all } \mathbf{T}$$

$$\leftrightarrow \quad \mathbf{C} \bullet (\mathbf{T} + \mathbf{S} \bullet \mathbf{T}) = \mathbf{S} \bullet \mathbf{C} \bullet (\mathbf{T} + \mathbf{S} \bullet \mathbf{T}) \quad \text{for all } \mathbf{T}$$

$$\leftrightarrow \quad \mathbf{C} \bullet (\mathbf{I} + \mathbf{S}) \bullet \mathbf{T} = \mathbf{S} \bullet \mathbf{C} \bullet (\mathbf{I} + \mathbf{S}) \bullet \mathbf{T} \quad \text{for all } \mathbf{T}$$

$$\leftrightarrow \quad \mathbf{C} + \mathbf{C} \bullet \mathbf{S} = \mathbf{S} \bullet \mathbf{C} + \mathbf{S} \bullet \mathbf{C} \bullet \mathbf{S}$$

□

**Theorem 2:**

For any tetrad $\mathbf{C}$, the conditions (S2) is equivalent to the condition

$$\mathbf{C} - \mathbf{C} \bullet \mathbf{S} = \mathbf{S} \bullet \mathbf{C} \bullet \mathbf{S} - \mathbf{S} \bullet \mathbf{C}$$

which has the component representation

$$c_{ijkl} - c_{ijlk} = c_{jilk} - c_{jikl} .$$

**Proof.** For any tetrad **C**, the condition (S2) is equivalent to

$$\mathbf{C} \bullet (\mathbf{I} - \mathbf{S}) \bullet \mathbf{T} = - \mathbf{S} \bullet \mathbf{C} \bullet (\mathbf{I} - \mathbf{S}) \bullet \mathbf{T} \quad \text{for all } \mathbf{T}$$
$$\leftrightarrow \mathbf{C} \bullet (\mathbf{I} - \mathbf{S}) = - \mathbf{S} \bullet \mathbf{C} \bullet (\mathbf{I} - \mathbf{S})$$
$$\leftrightarrow \mathbf{C} - \mathbf{C} \bullet \mathbf{S} = \mathbf{S} \bullet \mathbf{C} \bullet \mathbf{S} - \mathbf{S} \bullet \mathbf{C} \qquad \square$$

In order to show that the class of tetrads that satisfy (S1) and (S2), is not empty, we prove that the fourth-order identity **I** satisfies these conditions.

If $\mathbf{C} \equiv \mathbf{I}$

(S1) $\mathbf{I} + \mathbf{I} \bullet \mathbf{S} = \mathbf{I} + \mathbf{S} = \mathbf{S} + \mathbf{I} = \mathbf{S} \bullet \mathbf{I} + \mathbf{S} \bullet \mathbf{I} \bullet \mathbf{S}$

(S2) $\mathbf{I} - \mathbf{I} \bullet \mathbf{S} = \mathbf{I} - \mathbf{S} = \mathbf{S} \bullet \mathbf{I} \bullet \mathbf{S} - \mathbf{S} \bullet \mathbf{I}$

Clearly, each non-zero multiple of **I** also satisfies (S1) and (S2).

There remains the task to impose a normalization. We suggest the following:

**Normalization:** **C** is the identity for antisymmetric arguments:

$$\mathbf{B} = - \mathbf{B}^{t} \rightarrow \mathbf{C} \bullet \mathbf{B} = \mathbf{B}$$

Clearly, this does not affect the symmetric parts.

**Theorem 3:**

A tetrad **C** satisfies the normalization, if, and only if

(N) $$\mathbf{C} - \mathbf{C} \bullet \mathbf{S} = \mathbf{I} - \mathbf{S}$$

holds. The component form is

$$c_{ijkl} - c_{ijlk} = \delta_{ik}\, \delta_{jl} - \delta_{jk}\, \delta_{il} .$$

**Proof.** The normalization is equivalent to

$$\mathbf{C} \bullet (\mathbf{I} - \mathbf{S}) \bullet \mathbf{T} = (\mathbf{I} - \mathbf{S}) \bullet \mathbf{T} \quad \text{for all } \mathbf{T}$$
$$\leftrightarrow \mathbf{C} \bullet (\mathbf{I} - \mathbf{S}) = \mathbf{I} - \mathbf{S} \qquad \square$$

Evidently, $\mathbf{C} \equiv \mathbf{I}$ satisfies this condition , too. Hence, the set of tetrads, that satisfy (S1), (S2), (N), is not empty.

If $\mathbf{C}$ is invertible, the properties (S1) and (S2) automatically pass over to $\mathbf{C}^{-1}$. This is stated by the following

**Theorem 4:**

For any invertible tetrad $\mathbf{C}$ the conditions

| | |
|---|---|
| | (S1) for $\mathbf{C}$ |
| and | (S1) for $\mathbf{C}^{-1}$ |
| are equivalent, | |
| as well as | (S2) for $\mathbf{C}$ |
| and | (S2) for $\mathbf{C}^{-1}$ |

**Proof.** Let us call

LIN the of second-order tensors,
SYM the symmetric ones in LIN, and
SKW the antisymmetric ones in LIN.
SYM and SKW are linear subspaces of LIN, such that the only element in common is the tensor zero.
If we assume (S1) for $\mathbf{C}$, then

$$\mathbf{C}\,(\mathrm{SYM}) \subset \mathrm{SYM}$$

and, if $\mathbf{C}$ is invertible, $\mathbf{C}$ maps subspaces onto subspaces of the same dimension. Hence

$$\mathbf{C}\,(\mathrm{SYM}) = \mathrm{SYM}$$

and

$$\mathbf{C}^{-1}(\mathrm{SYM}) = \mathrm{SYM},$$

which proves (S1) for $\mathbf{C}^{-1}$.
The proof of the second part is strictly analogous. □

**Example**

The HOOKEan law as well as the NAVIER-STOKES law have the same LAME representation:

$$\mathbf{A} = \lambda\,(\mathbf{B} \bullet \mathbf{I})\,\mathbf{I} + 2\,\mu\,\mathbf{B}$$

where $\mathbf{I}$ is the second-order identity and $\lambda$ and $\mu$ are the LAME constants. In this case $\mathbf{A} = \mathbf{C} \bullet \mathbf{B}$

with

$$\mathbf{C} = \lambda\, \mathbf{I} \circ \mathbf{I} + 2\mu\, \mathbf{I}$$

satisfies (S1) and (S2) if $\mu$ is non-zero:

(S1): $\lambda\, \mathbf{I} \circ \mathbf{I} + 2\mu\, \mathbf{I} + \lambda\, \mathbf{I} \circ \mathbf{I} \bullet \mathbf{S} + 2\mu\, \mathbf{I} \bullet \mathbf{S} =$

$$\lambda\, \mathbf{S} \bullet \mathbf{I} \circ \mathbf{I} + 2\mu\, \mathbf{S} \bullet \mathbf{I} + \lambda\, \mathbf{S} \bullet \mathbf{I} \circ \mathbf{I} \bullet \mathbf{S} + 2\mu\, \mathbf{S} \bullet \mathbf{I} \bullet \mathbf{S}$$

(S2): $\lambda\, \mathbf{I} \circ \mathbf{I} + 2\mu\, \mathbf{I} - \lambda\, \mathbf{I} \circ \mathbf{I} \bullet \mathbf{S} - 2\mu\, \mathbf{I} \bullet \mathbf{S} =$

$$\lambda\, \mathbf{S} \bullet \mathbf{I} \circ \mathbf{I} \bullet \mathbf{S} + 2\mu\, \mathbf{S} \bullet \mathbf{I} \bullet \mathbf{S} - \lambda\, \mathbf{S} \bullet \mathbf{I} \circ \mathbf{I} - 2\mu\, \mathbf{S} \bullet \mathbf{I}$$

Entering the (N) condition, we obtain at the left-hand side
$\lambda\, \mathbf{I} \circ \mathbf{I} + 2\mu \mathbf{I} - \lambda\, \mathbf{I} \circ \mathbf{I} \bullet \mathbf{S} - 2\mu\, \mathbf{I} \bullet \mathbf{S} = 2\mu\, (\mathbf{I} - \mathbf{S})$

which equals the right-hand side $\mathbf{I} - \mathbf{S}$ if and only if $\mu = 1/2$. This result, clearly, is not acceptable, as it reduces the number of the free material parameters. To avoid this problem, we introduce

$$\mathbf{C} = \lambda\, \mathbf{I} \circ \mathbf{I} + 2\mu\, \mathbf{I} + (1/2 - \mu)\, (\mathbf{I} - \mathbf{S})$$

When $\mathbf{C}$ is acting on a symmetric tensor, the last part vanishes, whereas, when acting on an antisymmetric tensor $\mathbf{T}$, we obtain $\mathbf{C} \bullet \mathbf{T} = 0 + 2\mu\mathbf{T} + (1/2 - \mu)\, 2\, \mathbf{T} = \mathbf{T}$ and thus satisfies the normalization for any $\mu$, as well as (S1) and (S2) for any $\lambda$ and non-zero $\mu$. Moreover, $\mathbf{C}$ is symmetric:

$$\mathbf{A} \bullet \mathbf{C} \bullet \mathbf{B} = \lambda\, \mathrm{tr}\mathbf{A}\, \mathrm{tr}\mathbf{B} + 2\mu\mathbf{A} \bullet \mathbf{B} + (1/2 - \mu)\, (\mathbf{A} \bullet \mathbf{B} - \mathbf{A} \bullet \mathbf{B}^{t})$$

$$= \lambda\, \mathrm{tr}\mathbf{B}\, \mathrm{tr}\mathbf{A} + 2\mu\mathbf{B} \bullet \mathbf{A} + (1/2 - \mu)\, (\mathbf{B} \bullet \mathbf{A} - \mathbf{B} \bullet \mathbf{A}^{t})$$

$$= \mathbf{B} \bullet \mathbf{C} \bullet \mathbf{A}\,.$$

## 5. SECOND SUGGESTION

Another method of obtaining a linear mapping between (symmetric) tensors is used quite frequently in the literature. It makes use of the fact that symmetric tensors form a six-dimensional subspace of the tensor space, and one can regard its elements as six-dimensional vectors. This is done by means of the VOIGT-vectors

$$\mathbf{A}_v = (a_{11}, a_{22}, a_{33}, a_{12}, a_{23}, a_{31})$$
$$\mathbf{B}_v = (b_{11}, b_{22}, b_{33}, b_{12}, b_{23}, b_{31})$$

The tetrads can be represented by VOIGT-matrices as

$$\mathbf{C}_v = \begin{bmatrix} v_{11} & v_{12} & v_{13} & v_{14} & v_{15} & v_{16} \\ v_{21} & v_{22} & v_{23} & v_{24} & v_{25} & v_{26} \\ \dots & & & & & \\ v_{61} & v_{62} & v_{63} & v_{64} & v_{65} & v_{66} \end{bmatrix}$$

such that $\mathbf{A}_v = \mathbf{C}_v \,.\, \mathbf{B}_v$.

where the dot denotes the usual product between a (quadratic) matrice and a vector. In this way, any non-singular 6x6-matrice corresponds to exactly one non-singular tetrad that full fills (S1), (S2), and (N). Calculating two characteristic values of $\mathbf{A}_v$, f.e. $a_{11}$ and $a_{12}$ by the tetrade and the VOIGT-matrice, we obtain the identities:

$$\begin{aligned} a_{11} &= v_{11}\, b_{11} + v_{12}\, b_{22} + v_{13}\, b_{33} + v_{14}\, b_{12} + v_{15}\, b_{33} + v_{16}\, b_{31} \\ &= c_{1111}\, b_{11} + c_{1122}\, b_{22} + c_{1133}\, b_{33} \\ &\quad + (c_{1112} + c_{1121})\, b_{12} + (c_{1123} + c_{1132})\, b_{23} \\ &\quad + (c_{1131} + c_{1113})\, b_{31} \end{aligned}$$

$$\begin{aligned} a_{12} &= v_{41}\, b_{11} + v_{42}\, b_{22} + v_{43}\, b_{33} + v_{44}\, b_{12} + v_{45}\, b_{23} + v_{46}\, b_{31} \\ &= c_{1211}\, b_{11} \;\; c_{1222}\, b_{22} + c_{1233}\, b_{33} \\ &\quad + (c_{1212} + c_{1221})\, b_{12} + (c_{1223} + c_{1232})\, b_{23} \\ &\quad + (c_{1231} + c_{1213})\, b_{31} \end{aligned}$$

Comparing the coefficents of $b_{ij}$, we obtain the VOIGT-matrice

$$\mathbf{C}_v = \begin{bmatrix} c_{1111} & c_{1122} & c_{1133} & c_{1112} + c_{1121} & c_{1123} + c_{1132} & c_{1131} + c_{1113} \\ c_{2211} & c_{2222} & c_{2233} & c_{2212} + c_{2221} & c_{2223} + c_{2232} & c_{2231} + c_{2213} \\ c_{3311} & c_{3322} & c_{3333} & c_{3312} + c_{3321} & c_{3323} + c_{3332} & c_{3331} + c_{3313} \\ c_{1211} & c_{1222} & c_{1233} & c_{1212} + c_{1221} & c_{1223} + c_{1232} & c_{1231} + c_{1213} \\ c_{2311} & c_{2322} & c_{2333} & c_{2312} + c_{2321} & c_{2323} + c_{2332} & c_{2331} + c_{2313} \\ c_{3111} & c_{3122} & c_{3133} & c_{3112} + c_{3121} & c_{3123} + c_{3132} & c_{3131} + c_{3113} \end{bmatrix}$$

which can be simplified by means of the normalization in the form

$$c_{ijkl} = \delta_{ik}\,\delta_{jl} - \delta_{jk}\,\delta_{il} + c_{ijlk}$$

$$
C_v = \begin{bmatrix}
c_{1111} & c_{1122} & c_{1133} & 2.c_{1112} & 2.c_{1123} & 2.c_{1131} \\
c_{2211} & c_{2222} & c_{2233} & 2.c_{2212} & 2.c_{2223} & 2.c_{2231} \\
c_{3311} & c_{3322} & c_{3333} & 2.c_{3312} & 2.c_{3323} & 2.c_{3331} \\
c_{1211} & c_{1222} & c_{1233} & 2.c_{1212}-1 & 2.c_{1223} & 2.c_{1231} \\
c_{2311} & c_{2322} & c_{2333} & 2.c_{2312} & 2.c_{2323}-1 & 2.c_{2331} \\
c_{3111} & c_{3122} & c_{3133} & 2.c_{3112} & 2.c_{3123} & 2.c_{3131}-1
\end{bmatrix}
$$

**Example:** In continuation of the previous example we obtain the following VOIGT-matrice for the LAME -type tetrad with

$$c_{ijkl} = \lambda\,\delta_{ij}\,\delta_{kl} + 2\mu\delta_{ik}\,\delta_{jl} + (1/2 - \mu)(\delta_{ik}\,\delta_{jl} - \delta_{il}\,\delta_{jk})$$

$$
C_v = \begin{bmatrix}
\lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\
\lambda & \lambda + 2\mu & 0 & 0 & 0 & 0 \\
\lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\
0 & 0 & 0 & 2\mu & 0 & 0 \\
0 & 0 & 0 & 0 & 2\mu & 0 \\
0 & 0 & 0 & 0 & 0 & 2\mu
\end{bmatrix}
$$

## 6. ADDITIONAL SYMMETRIES

In many applications there exist physical facts (conservativity, etc.) that impose the symmetry of **C**. This is completely independent of the conditions (S1), (S2), and (N).

The component form of the symmetry of the tetrad **C** is

$$c_{ijkl} = c_{klij}\,,$$

whereas the corresponding relations for the VOITG-matrice are

$$
\begin{array}{llll}
v_{12} = v_{21} & 2.v_{14} = v_{41} & 2.v_{24} = v_{42} & 2.v_{34} = v_{43} \\
v_{23} = v_{32} & 2.v_{15} = v_{51} & 2.v_{25} = v_{52} & 2.v_{35} = v_{53} \\
v_{31} = v_{13} & 2.v_{16} = v_{61} & 2.v_{26} = v_{62} & 2.v_{36} = v_{63} \\
v_{45} = v_{54} & 2.v_{45} = v_{54} & & \\
v_{56} = v_{65} & 2.v_{56} = v_{65} & & \\
v_{64} = v_{46} & 2.v_{64} = v_{46} & &
\end{array}
$$

This means that the symmetry of **C** and the symmetry of $\mathbf{C}_v$ are not equivalent. In order to avoid this unpleasant situation, we introduce the VOIGT-vectors and the VOIGT-matrices in the following manner:

$$\mathbf{A}_v = (x\, a_{11}\, ,\, x\, a_{22}\, ,\, x\, a_{33}\, ,\, y\, a_{12}\, ,\, y\, a_{23}\, ,\, y\, a_{31})$$

$$\mathbf{B}_v = (x\, b_{11}\, ,\, x\, b_{22}\, ,\, x\, b_{33}\, ,\, y\, b_{12}\, ,\, y\, b_{23}\, ,\, y\, b_{31})$$

with two real factors x and y.

The corresponding VOIGT-matrice is

$$\mathbf{C}_v = \begin{bmatrix} c_{1111} & c_{1122} & c_{1133} & 2.c_{1112}\, x/y & 2.c_{1123}\, x/y & 2.c_{1131}\, x/y \\ c_{2211} & c_{2222} & c_{2233} & 2.c_{2212}\, x/y & 2.c_{2223}\, x/y & 2.c_{2231}\, x/y \\ c_{3311} & c_{3322} & c_{3333} & 2.c_{3312}\, x/y & 2.c_{3323}\, x/y & 2.c_{3331}\, x/y \\ c_{1211}\, x/y & c_{1222}\, x/y & c_{1233}\, x/y & 2.c_{1212} - 1 & 2.c_{1223} & 2.c_{1231} \\ c_{2311}\, x/y & c_{2322}\, x/y & c_{2333}\, x/y & 2.c_{2312} & 2.c_{2323} - 1 & 2.c_{2331} \\ c_{3111}\, x/y & c_{3122}\, x/y & c_{3133}\, x/y & 2.c_{3112} & 2.c_{3123} & 2.c_{3131} - 1 \end{bmatrix}$$

A small calculus shows, that its symmetry coincides with the symmetry of **C**, if it holds for the factors that

$$2\, x^2 = y^2 .$$

This can be easily fulfilled by taking

$$x = 1 \qquad \text{and} \qquad y = \sqrt{2}\, ,$$

a well-known method.

## 7. THE COMPONENT FORMS OF THE CONDITIONS FOR C

The tetrad **C** based on a three-dimensional vector space has in general 3x3x3x3 = 81 components.

### The (S1) condition

has the following component form:

$$c_{2111} - c_{1211} = 0 \qquad c_{2121} + c_{2112} - c_{1221} - c_{1212} = 0$$
$$c_{3111} - c_{1311} = 0 \qquad c_{3121} + c_{3112} - c_{1321} - c_{1312} = 0$$
$$c_{3211} - c_{2311} = 0 \qquad c_{3221} + c_{3212} - c_{2321} - c_{2312} = 0$$

$$
\begin{array}{ll}
c_{2122} - c_{1222} = 0 & c_{2131} + c_{2113} - c_{1231} - c_{1213} = 0 \\
c_{3122} - c_{1322} = 0 & c_{3131} + c_{3113} - c_{1331} - c_{1313} = 0 \\
c_{3222} - c_{2322} = 0 & c_{3231} + c_{3213} - c_{2331} - c_{2313} = 0 \\
c_{2133} - c_{1233} = 0 & c_{2132} + c_{2123} - c_{1232} - c_{1223} = 0 \\
c_{3133} - c_{1333} = 0 & c_{3132} + c_{3123} - c_{1332} - c_{1323} = 0 \\
c_{3233} - c_{2333} = 0 & c_{3232} + c_{3223} - c_{2332} - c_{2323} = 0
\end{array}
$$

These are 18 independent conditions.

**The (S2) condition**

$$
\begin{array}{ll}
c_{1121} - c_{1112} = 0 & c_{2121} + c_{1221} - c_{2112} - c_{1212} = 0 \\
c_{2221} - c_{2212} = 0 & c_{3121} + c_{1321} - c_{3112} - c_{1312} = 0 \\
c_{3321} - c_{3312} = 0 & c_{3221} + c_{2321} - c_{3212} - c_{2312} = 0 \\
c_{1131} - c_{1113} = 0 & c_{2131} + c_{1231} - c_{2113} - c_{1213} = 0 \\
c_{2231} - c_{2213} = 0 & c_{3131} + c_{1321} - c_{3113} - c_{1313} = 0 \\
c_{3331} - c_{3313} = 0 & c_{3231} + c_{2331} - c_{3213} - c_{2313} = 0 \\
c_{1132} - c_{1123} = 0 & c_{2132} + c_{1232} - c_{2123} - c_{1223} = 0 \\
c_{2232} - c_{2223} = 0 & c_{3132} + c_{1332} - c_{3123} - c_{1323} = 0 \\
c_{3332} - c_{3323} = 0 & c_{3232} + c_{2332} - c_{3223} - c_{2323} = 0
\end{array}
$$

These are also 18 independent conditions.

**The normalization**

$$
\begin{array}{lll}
c_{2121} - c_{2112} = 1 & c_{3131} - c_{3113} = 1 & c_{3232} - c_{3223} = 1 \\
c_{1221} - c_{1212} = -1 & c_{1331} - c_{1313} = -1 & c_{2332} - c_{2323} = -1 \\
c_{1121} - c_{1112} = 0 & c_{1131} - c_{1113} = 0 & c_{1132} - c_{1123} = 0 \\
c_{3121} - c_{3112} = 0 & c_{2131} - c_{2113} = 0 & c_{2132} - c_{2123} = 0 \\
c_{2221} - c_{2212} = 0 & c_{1231} - c_{1213} = 0 & c_{3132} - c_{3123} = 0 \\
c_{3221} - c_{3212} = 0 & c_{2231} - c_{2213} = 0 & c_{1232} - c_{1223} = 0 \\
c_{1321} - c_{1312} = 0 & c_{3231} - c_{3213} = 0 & c_{2232} - c_{2223} = 0 \\
c_{2321} - c_{2312} = 0 & c_{2331} - c_{2313} = 0 & c_{1332} - c_{1323} = 0 \\
c_{3321} - c_{3312} = 0 & c_{3331} - c_{3313} = 0 & c_{3332} - c_{3323} = 0
\end{array}
$$

These are 27 independent conditions. 9 of them are identical with those of (S2). Moreover, there exist 9 groups of 4 dependent equations from (S1) and (S2) and (N), such that there finally remain 36 independent components.

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**RESUMEN**

**En muchas aplicaciones físicas y técnicas se necesita funciones lineales entre tensores simétricos. Se muestran sus propiedades y representaciones para implementar tales funciones en el conjunto de tensores del cuarto orden, evitando desventajas de los métodos corrientes.**

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