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**Abstract**

The evolution of material damage of single crystal superalloys depends not only on the load conditions, but also strongly on the lattice orientation. Using the theory of continuum damage mechanics, a phenomenological creep damage model for cubic single crystal superalloys is derived. In this model, a symmetric second-order damage tensor is used to describe the anisotropic nature of damage. The damage deactivation and reactivation is represented by an active damage tensor. As the effects of the current state of damage on the deformation process and on the damage development are different in nature, separate effective stresses for creep and damage are defined. Furthermore, a mapped damage active stress is introduced to reflect the influence of material symmetry on the damage growth rate by using an orientation function. Based on microscopic observations, it is assumed that only the principle tensile damage active stresses are responsible for the damage development, and that the anisotropy of the damage evolution mainly depends on the principle directions of the damage active stress and the material symmetry. Within the framework of thermodynamics, the damage evolution law is constructed by considering both the initial material anisotropy and the anisotropic influence of the current state of damage. Based on the effective stress concept, this damage model has been implemented into a three-dimensional anisotropic viscoplastic model and applied to the simulation of the creep damage behavior of the single crystal superalloy SRR99 at 760°C. 

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*Keywords:* Creep damage; Anisotropic damage; Constitutive modeling; Continuum damage mechanics; Single crystal superalloys

## 1. Introduction

Nickel-based superalloys in their monocrystalline form are widely used in components of heat engines such as turbine blades, where it is desirable to keep creep deformations to a minimum. In order to take full advantage of the superior high temperature strength of these materials, experimental and theoretical investigations of creep deformations and damage of single crystal superalloys are necessary.

Mechanical properties of single crystal superalloys at elevated temperatures are strongly anisotropic and nonlinear [1–7]. Experimental and microscopic investigations show that the creep behavior of single crystal superalloys is influenced by the lattice orientation, and that the material damage is directly caused by the growth and coalescence of initial microcracks starting from

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imperfections such as casting pores. Thus, the damage develops anisotropically. Therefore there is a need to develop constitutive models for the growth of creep strain and damage of single crystal superalloys by considering the influence of material and damage induced anisotropy. These models should on the one hand adequately represent material behavior, but, on the other, be as simple as possible so that they are in a form suitable for implementation into numerical codes such as FEM.

Advanced engineering materials subjected to extreme thermo-mechanical and environmental conditions undergo microstructural changes which reduce their deformation resistance. Bertram and Olschewski [5] proposed an anisotropic constitutive model for the description of the creep behavior of single crystals at high temperatures. This model has been applied to the creep behavior simulation of the single crystal superalloy SRR99 [6] and of CMSX-6 [7] at 760°C. However, the progressive degradation of the mechanical properties of such materials caused by irreversible microstructural changes during the creep process has not been considered in this approach. It is therefore restricted to the undamaged material behavior of the primary and secondary creep phase. In order to model the tertiary creep behavior and to predict the time to rupture, additional models are needed.

Continuum Damage Mechanics (CDM) as developed by Kachanov [8,9], Rabotnov [10], Hult [11], Krajcinovic and Lemaitre [12], Chaboche [13], Lemaitre [14] and others is a continuum mechanics approach to the deformation and rupture analysis. CDM concepts are supported by the general framework of thermodynamics of irreversible processes, and offer complementary possibilities to fracture mechanics [13]. In this paper, we will first briefly introduce the creep model for single crystal superalloys proposed by Bertram and Olschewski [5–7]. The focus will be on a phenomenological damage model developed on the grounds of the CDM theory by considering both the initial anisotropy of the material and the damage induced anisotropy, and on its application to the single crystal SRR99 at 760°C by using the effective stress concept of CDM. As such materials have limited ductility, we apply only small deformation analysis. For simplicity, isothermal conditions are assumed so that the effect of temperature changes enters the constitutive equations only through the temperature dependence of the material parameters.

## 2. Creep model for single crystals

Starting from a uniaxial rheological four-parameter Burgers-model which consists of two springs and two dampers, Bertram and Olschewski [5] used a projection method to construct an anisotropic three-dimensional model for the description of the creep behavior of cubic single crystals at high temperatures. These constitutive equations can be written as:

$$\dot{\boldsymbol{\varepsilon}} = \overset{\langle 4\rangle}{\mathbf{A}}_1 : \dot{\boldsymbol{\sigma}} + \overset{\langle 4\rangle}{\mathbf{A}}_2 : \boldsymbol{\sigma} + \overset{\langle 4\rangle}{\mathbf{A}}_3 : \boldsymbol{\tau}, \tag{1}$$

$$\dot{\boldsymbol{\tau}} = \overset{\langle 4\rangle}{\mathbf{A}}_4 : \dot{\boldsymbol{\sigma}} + \overset{\langle 4\rangle}{\mathbf{A}}_5 : (\boldsymbol{\sigma} - \boldsymbol{\tau}), \tag{2}$$

where $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ are the strain and stress tensor, respectively, and $\boldsymbol{\tau}$ is an internal variable which corresponds, in uniaxial cases, to the stress in a spring. Herein

$$\overset{\langle 4\rangle}{\mathbf{A}}_1 = \sum_{i=1}^{3} \frac{1}{C_i + K_i} \overset{\langle 4\rangle}{\mathbf{P}}_i, \tag{3a}$$

$$\overset{\langle 4\rangle}{\mathbf{A}}_2 = \sum_{i=1}^{3} \frac{1}{C_i + K_i} \left( \frac{C_i}{D_i} + \frac{C_i}{L_i} + \frac{K_i}{L_i} \right) \overset{\langle 4\rangle}{\mathbf{P}}_i, \tag{3b}$$

$$\overset{\langle 4\rangle}{\mathbf{A}}_3 = \sum_{i=1}^{3} \frac{C_i}{C_i + K_i} \frac{1}{D_i} \overset{\langle 4\rangle}{\mathbf{P}}_i, \tag{3c}$$

$$\overset{\langle 4\rangle}{\mathbf{A}}_4 = \sum_{i=1}^{3} \frac{K_i}{C_i + K_i} \overset{\langle 4\rangle}{\mathbf{P}}_i, \tag{3d}$$

$$\overset{\langle 4\rangle}{\mathbf{A}}_5 = \sum_{i=1}^{3} \frac{K_i}{C_i + K_i} \frac{C_i}{D_i} \overset{\langle 4\rangle}{\mathbf{P}}_i \tag{3e}$$

are fourth-rank material tensors being linear combinations of three structure tensors

$$\overset{\langle 4\rangle}{\mathbf{P}}_1 = \frac{1}{3}\mathbf{I}\otimes\mathbf{I}, \tag{4a}$$

$$\overset{\langle 4\rangle}{\mathbf{P}}_2 = \overset{\langle 4\rangle}{\mathbf{R}} - \overset{\langle 4\rangle}{\mathbf{P}}_1, \tag{4b}$$

$$\overset{\langle 4\rangle}{\mathbf{P}}_3 = \overset{\langle 4\rangle}{\mathbf{I}} - \overset{\langle 4\rangle}{\mathbf{P}}_1 - \overset{\langle 4\rangle}{\mathbf{P}}_2, \tag{4c}$$

with

$$\overset{\langle 4\rangle}{\mathbf{R}} = \sum_{i=1}^{3}\mathbf{e}_i^k\otimes\mathbf{e}_i^k\otimes\mathbf{e}_i^k\otimes\mathbf{e}_i^k, \tag{5}$$

where $\mathbf{I}$ and $\overset{\langle 4\rangle}{\mathbf{I}}$ denote the identity tensor of rank two and four, respectively, $\otimes$ the tensor product, $C_i$, $K_i$, $D_i$, $L_i$ $(i = 1,2,3)$ temperature-dependent material parameters, and $\mathbf{e}_j^k$ $(j = 1,2,3)$ the lattice vectors. The dependence of the viscosities $D_i$ and $L_i$ on the applied stress is expressed as:

$$D_i = D_{0i}\,\exp\left\{-\sum_j Z_{ij}J_j\right\}, \tag{6a}$$

$$L_i = L_{0i}\,\exp\left\{-\sum_j Z_{ij}J_j\right\}, \tag{6b}$$

with material parameters $Z_{ij}$ $(i = 1,2,3,4;\ j = 1,2,3)$ and the following scalar invariants showing cubic symmetry

$$J_1 = \sqrt{\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11}}, \tag{7a}$$

$$J_2 = \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2, \tag{7b}$$

$$J_3 = \sigma_{11}\sigma_{22}\sigma_{33}, \tag{7c}$$

$$J_4 = \sigma_{11}(\sigma_{12}^2 + \sigma_{13}^2) + \sigma_{22}(\sigma_{23}^2 + \sigma_{21}^2) + \sigma_{33}(\sigma_{31}^2 + \sigma_{32}^2). \tag{7d}$$

By the assumption that volume changes occur only elastically, it follows:

$$D_1^{-1} = 0,\ L_1^{-1} = 0 \quad\text{and}\quad Z_{i1} = 0\ (i = 1,\ 2,\ 3,\ 4). \tag{8}$$

In the limiting cases, $D_i = \infty$ and $L_i = \infty$ $(i = 1,2,3)$, the Eqs. (1) and (2) degenerate to Hooke's law.

## 3. The effective stress concept of CDM

Damage resulting from the development of microcracks and microvoids leads to creep acceleration in the tertiary creep phase. This effect can be described by adding the damage variables into the creep equation. As the degradation of the material structure caused by material damage implies an internal decrease of the load carrying area, Rabotnov [10] first introduced the notion of effective stress

$$\tilde{\sigma} = \frac{\sigma}{1-D}, \tag{9}$$

where $\sigma$ is the stress and $D$ is the damage variable, and modified Norton's steady stage creep law

$$\dot{\varepsilon}^c = A\sigma^w \tag{10}$$

by replacing the stress $\sigma$ by the effective stress $\tilde{\sigma}$, i.e.

$$\dot{\varepsilon}^c = A\tilde{\sigma}^w = A\left(\frac{\sigma}{1-D}\right)^w, \tag{11}$$

where $\varepsilon^c$ is the creep strain, and $A$ and $w$ are material parameters. The following evolution law of damage is then assumed to complete the model:

$$\dot{D} = G\left(\frac{\sigma}{1-D}\right)^r. \tag{12}$$

Two material parameters $G$ and $r$ are to be determined from creep tests. This model can describe fairly accurately the tertiary creep as well as the creep ductility in many materials in the one-dimensional case [15].

Following this idea it can be assumed that there exists a fictitious effective stress acting on the undamaged material that causes the same strains as the real stress acting on the damaged material. The effective stress concept has been expressed by Lemaitre [14]: "Any strain constitutive equation for a damaged material may be derived in the same way as for a virgin material except that the usual stress is replaced by the effective stress".

## 4. Phenomenological damage model

Microscopic investigations [16–18] show that the deterioration of nickel-based single crystal superalloys under creep loading conditions is directly caused by the growth and coalescence of initial microcracks starting from casting pores. Material damage can be defined as a collection of permanent microstructural changes concerning the material thermomechanical properties (e.g. stiffness, strength, anisotropy, etc.) brought about in a material by irreversible physical microcracking processes resulting from the application of thermomechanical loading [19,20]. According to the principles of irreversible thermodynamics it is necessary to introduce internal variables, called damage variables, for the phenomenological description of material damage. Because of its microscopic nature damage has, in general, an anisotropic character even if the material is originally isotropic [21]. Therefore, tensor valued variables must be used for the three-dimensional representation of material damage [22]. In order to represent the state of anisotropic damage characterized by these cracks, we will employ a second rank symmetric damage tensor $\mathbf{D}$ as a damage variable.

According to the effective stress concept, the classical continuum mechanics model for the undamaged material, such as the one described in Part 2, can be modified to include the damage influence. To accomplish this, a suitable definition of effective stress, which will replace the stress tensor in Eqs. (1) and (2), and a damage evolution law are needed.

### 4.1. Effective stress and damage deactivation

Let a fourth-order tensor $\overset{(4)}{\mathbf{M}}(\mathbf{D})$ characterize the damage state. The following general form of the transformation between the stress tensor $\boldsymbol{\sigma}$ and the effective stress tensor $\tilde{\boldsymbol{\sigma}}$ is assumed [15,23]:

$$\tilde{\boldsymbol{\sigma}} = \overset{(4)}{\mathbf{M}}(\mathbf{D}) : \boldsymbol{\sigma}. \tag{13}$$

Following the suggestion of Cordebois and Sidoroff [24], the previous transformation is taken in the particular form

$$\overset{(4)}{\mathbf{M}} = (\mathbf{I}-\mathbf{D})^{-1/2} \wedge (\mathbf{I}-\mathbf{D})^{-1/2}, \tag{14}$$

where the composition of two second-order tensors denoted by the wedge $\wedge$ is defined as $\mathbf{A} \wedge \mathbf{B} = a_{ij}b_{kl}(\mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_j \otimes \mathbf{e}_l)$ for an orthonormal basis $\mathbf{e}_i$. This suggestion for $\overset{(4)}{\mathbf{M}}(\mathbf{D})$ is identical to the one of Chow and Wang [25,26], and has been applied to the dynamic fracture of brittle anisotropic solids [27] and to the analytical prediction of the initiation and propagation of ductile fracture in metals [28]. However, it does not consider the deactivation of damage due to crack closure.

In order to illustrate the assumed damage process, we consider a single crack embedded in an elastic material with a tensile load perpendicular to the crack. If the load is reversed, the crack will close and in a one-dimensional case the material behaves as uncracked. This phenomenon is called damage deactivation (not healing) in CDM. The damage still exists but the loading condition can render it inactive. For the representation of this mechanism the phenomenological algorithm proposed by Hansen and Schreyer [29] is used in the present model. In this method the microcrack opening/closing effect is described by considering the spectral decomposition of the elastic strain tensor $\boldsymbol{\varepsilon}^e$ and the total strain tensor $\boldsymbol{\varepsilon}$

$$\boldsymbol{\varepsilon}^e = \sum_{i=1}^{3} \varepsilon_i^e \mathbf{n}_{ei}^{\varepsilon} \otimes \mathbf{n}_{ei}^{\varepsilon}, \tag{15a}$$

$$\boldsymbol{\varepsilon} = \sum_{i=1}^{3} \varepsilon_i \mathbf{n}_i^{\varepsilon} \otimes \mathbf{n}_i^{\varepsilon}, \tag{15b}$$

where $\varepsilon_i^e$ and $\varepsilon_i$ are the $i$th eigenvalues, $\mathbf{n}_{ei}^{\varepsilon}$ and $\mathbf{n}_i^{\varepsilon}$ are the corresponding ith eigenvectors of $\boldsymbol{\varepsilon}^e$ and $\boldsymbol{\varepsilon}$, respectively. The positive (tensile) spectral tensor corresponding to the elastic and to the total strain are defined as

$$\mathbf{H}_{\varepsilon e} = \sum_{i=1}^{3} h(\varepsilon_i^e) \mathbf{n}_{ei}^{\varepsilon} \otimes \mathbf{n}_{ei}^{\varepsilon}, \tag{16a}$$

$$\mathbf{H}_{\varepsilon} = \sum_{i=1}^{3} h(\varepsilon_i) \boldsymbol{n}_i^{\varepsilon} \otimes \mathbf{n}_i^{\varepsilon}, \tag{16b}$$

respectively, where $h(x)$ is the Heaviside function

$$h(x) = \begin{cases} 0 & \text{for } x \leqslant 0, \\ 1 & \text{for } x > 0, \end{cases} \tag{17a}$$

or, if one wants to avoid the discontinuity of stress–strain response, the following smooth function proposed in [29] can be used:

$$h(x) = \begin{cases} 0 & \text{for } x \leqslant x_m, \\ \frac{1}{2}\left\{1 - \cos\left[\frac{\pi(x - x_m)}{x_p - x_m}\right]\right\} & \text{for } x_m < x < x_p, \\ 1 & \text{for } x \geqslant x_p, \end{cases} \tag{17b}$$

where $x_m$ and $x_p$ are two additional parameters. The positive spectral projection operators (fourth-order tensor) for the elastic and the total strains are defined as

$$\overset{\langle 4\rangle}{\mathbf{P}}_{\varepsilon e} = \mathbf{H}_{\varepsilon e} \wedge \mathbf{H}_{\varepsilon e}, \tag{18a}$$

$$\overset{\langle 4\rangle}{\mathbf{P}}_{\varepsilon} = \mathbf{H}_{\varepsilon} \wedge \mathbf{H}_{\varepsilon}, \tag{18b}$$

respectively. The positive projection of the elastic and the total strain tensors are then given by

$$\boldsymbol{\varepsilon}^{e+} = \overset{\langle 4\rangle}{\mathbf{P}}_{\varepsilon e} \colon \boldsymbol{\varepsilon}^{e}, \tag{19a}$$

$$\boldsymbol{\varepsilon}^{+} = \overset{\langle 4\rangle}{\mathbf{P}}_{\varepsilon} \colon \boldsymbol{\varepsilon}, \tag{19b}$$

respectively, where the symbol: denotes the double contraction. By introducing a strain-based positive projection operator

$$\overset{\langle 4\rangle}{\mathbf{T}} = \overset{\langle 4\rangle}{\mathbf{I}} - \left(\overset{\langle 4\rangle}{\mathbf{I}} - \overset{\langle 4\rangle}{\mathbf{P}}_{\varepsilon e}\right) \colon \left(\overset{\langle 4\rangle}{\mathbf{I}} - \overset{\langle 4\rangle}{\mathbf{P}}_{\varepsilon}\right) \tag{20}$$

the *active damage* tensor is then given by

$$\mathbf{D}_a = \overset{\langle 4\rangle}{\mathbf{T}} \colon \mathbf{D}. \tag{21}$$

Thus, the *effective stress tensor* with respect to damage deactivation is defined as

$$\tilde{\boldsymbol{\sigma}} = (1 - \mathbf{D}_a)^{-1/2} \cdot \boldsymbol{\sigma} \cdot (1 - \mathbf{D}_a)^{-1/2}. \tag{22}$$

### *4.2. Damage evolution equation*

By assuming a decoupling of intrinsic and thermal dissipation and, for practical purposes, by postulating that the dissipation due to the damage processes and the dissipation associated with the other mechanisms such as the plastic strain and the hardening process, are independent [30], the dual dissipation potential $\phi$ can be written as

$$\phi = \phi_{\mathrm{T}} + \phi_{\mathrm{M}} + \phi_{\mathrm{D}}, \tag{23}$$

where $\phi_{\mathrm{T}}$, $\phi_{\mathrm{D}}$ and $\phi_{\mathrm{M}}$ are the dual potentials associated with thermal, damage, and other mechanisms, respectively. Since

$$\frac{\partial \phi_{\mathrm{T}}}{\partial \mathbf{Y}_{\mathrm{D}}} = 0, \tag{24a}$$

$$\frac{\partial \phi_{\mathrm{M}}}{\partial \mathbf{Y}_{\mathrm{D}}} = 0 \tag{24b}$$

the damage evolution law is given in [31,32]

$$\dot{\mathbf{D}} = \frac{\partial \phi}{\partial \mathbf{Y}_{\mathrm{D}}} = \frac{\partial \phi_{\mathrm{D}}}{\partial \mathbf{Y}_{\mathrm{D}}}, \tag{24c}$$

where $\mathbf{Y}_{\mathrm{D}}$ is the thermodynamic force associated with damage, called *damage driving force*. To avoid complicated numerical computation, a quadratic form is chosen for the expression of the damage dissipation potential

$$\phi_{\mathrm{D}} = \frac{1}{2} \mathbf{Y}_{\mathrm{D}} \colon \overset{\langle 4\rangle}{\mathbf{S}} \colon \mathbf{Y}_{\mathrm{D}}, \tag{25}$$

where the fourth-order *structure tensor* $\overset{\langle 4\rangle}{\mathbf{S}}$ must be symmetric and positive-definite (thermodynamic restrictions). The damage evolution law is then given by

$$\dot{\mathbf{D}} = \overset{\langle 4\rangle}{\mathbf{S}} \colon \mathbf{Y}_{\mathrm{D}}. \tag{26}$$

In order to simplify the problem, we consider at first isotropic materials. In this case, damage development in a creep process generally depends on the current state of stress and damage. The damage law can be expressed as

$$\dot{\mathbf{D}} = \mathbf{G}(\boldsymbol{\sigma}, \mathbf{D}). \tag{27}$$

Analogous to the effective stress, which represents the effect of stress and damage on the strain response, a *damage active stress* $\hat{\boldsymbol{\sigma}}$ can be assumed to represent the contributions of both the stress and the damage state to the damage growth. As experimental observations show that the creep rate is less sensitive to the damage state in comparison with the rate of void growth [33], the damage active stress is defined as

$$\hat{\boldsymbol{\sigma}} = (1 - \mathbf{D}_a)^{-p} \cdot \boldsymbol{\sigma} \cdot (1 - \mathbf{D}_a)^{-p}, \tag{28}$$

with a material parameter $p$ which is used to distinguish the effect of damage on the damage growth from that on the creep rate. Thus, the damage development can be assumed to be only dependent on the damage active stress. On the other hand, according to thermodynamics, the damage driving force is responsible for the damage development. Based on the results of the microscopic investigations we assume that only the tensile damage active stresses are responsible for the damage growth, and that for isotropic materials the anisotropy of damage development only depends on the principal directions of damage active stress tensor. Motivated by these considerations and the uniaxial damage model of Kachanov and Rabotnov, we postulate the following expression for the damage driving force for isotropic materials

$$\mathbf{Y}_\mathrm{D} = \langle \hat{\boldsymbol{\sigma}} \rangle^n = \sum_{i=1}^{3} \langle \hat{\sigma}_i \rangle^n \mathbf{n}_i^\sigma \otimes \mathbf{n}_i^\sigma, \tag{29}$$

where $n$ is a material parameter, $\hat{\sigma}_i$ and $\hat{\mathbf{n}}_i^\sigma$ are the $i$th eigenvalue and eigenvector of $\hat{\boldsymbol{\sigma}}$, and $\langle . \rangle$ is the McCauley bracket, which equals one for positive arguments and zero otherwise.

For an anisotropic material, however, the damage will develop differently if the same tensile stress acts in different directions. In order to obtain the creep potential of anisotropic solids, Betten [34] proposed to map the actual creep state of an anisotropic solid onto a fictitious isotropic state with equivalent creep rate by a suitable transformation. The anisotropic behavior is described by using a *mapped stress tensor* instead of the actual stress tensor in the isotropic creep potential. Following this idea, we introduce the *mapped damage active stress* $\hat{\boldsymbol{\sigma}}_m$ with the help of an orientation function $\boldsymbol{\eta}_i$

$$\hat{\boldsymbol{\sigma}}_m = \sum_{i=1}^{3} (\eta_i \hat{\sigma}_i) \hat{\mathbf{n}}_i^\sigma \otimes \hat{\mathbf{n}}_i^\sigma. \tag{30}$$

The orientation function $\eta_i$ modifies the effect of the $i$th principal damage active stress $\hat{\sigma}_i$ depending on its orientation related to the material structure. For single crystals with cubic symmetry we suggest the following orientation function:

$$\eta_i = \left[ \sum_{j=1}^{3} \left( \hat{\mathbf{n}}_i^\sigma \, \mathbf{e}_j^k \right)^{2m} \right]^q \tag{31}$$

where $m$ and $q$ are material parameters, and $\mathbf{e}_j^k$ $(j = 1, 2, 3)$ are the lattice vectors. The damage driving force (with respect to material symmetry) is then given by

$$\mathbf{Y}_D = \sum_{i=1}^{3} \langle \eta_i \hat{\sigma}_i \rangle^n \hat{\mathbf{n}}_i^\sigma \otimes \hat{\mathbf{n}}_i^\sigma. \tag{32}$$

For cubic single crystals, we consider the following expression of the structure tensor $\overset{\langle 4 \rangle}{\mathbf{S}}$ (see Eqs. (4a)–(4c))

$$\overset{\langle 4 \rangle}{\mathbf{S}} = \xi_1 \overset{\langle 4 \rangle}{\mathbf{P}}_1 + \xi_2 \overset{\langle 4 \rangle}{\mathbf{P}}_2 + \xi_3 \overset{\langle 4 \rangle}{\mathbf{P}}_3, \tag{33}$$

where $\xi_1$, $\xi_2$ and $\xi_3$ are material parameters. From Eqs. (4a)–(4c) we obtain

$$\overset{\langle 4 \rangle}{\mathbf{S}} = \beta_1 \mathbf{I} \otimes \mathbf{I} + \beta_2 \overset{\langle 4 \rangle}{\mathbf{I}} + \beta_3 \overset{\langle 4 \rangle}{\mathbf{R}}, \tag{34}$$

with

$$\beta_1 = \frac{\xi_1 - \xi_2}{3}, \quad \beta_2 = \xi_2, \quad \beta_3 = \xi_2 - \xi_3. \tag{35}$$

Substituting Eqs. (32) and (34) into Eq. (26), we obtain the damage evolution law

$$\dot{\mathbf{D}} = \left( \beta_1 \mathbf{I} \otimes \mathbf{I} + \beta_2 \overset{\langle 4 \rangle}{\mathbf{I}} + \beta_3 \overset{\langle 4 \rangle}{\mathbf{R}} \right) : \sum_{i=1}^{3} \langle \eta_i \hat{\sigma}_i \rangle^n \hat{\mathbf{n}}_i^\sigma \otimes \hat{\mathbf{n}}_i^\sigma \tag{36}$$

or, alternatively

$$\dot{\mathbf{D}} = \left( \alpha_1 \mathbf{I} \otimes \mathbf{I} + \alpha_2 \overset{\langle 4 \rangle}{\mathbf{I}} + \alpha_3 \overset{\langle 4 \rangle}{\mathbf{R}} \right) : \sum_{i=1}^{3} \left\langle \frac{\eta_i \hat{\sigma}_i}{B} \right\rangle^n \hat{\mathbf{n}}_i^\sigma \otimes \hat{\mathbf{n}}_i^\sigma, \tag{37}$$

with

$$\alpha_1 + \alpha_2 + \alpha_3 = 1. \tag{38}$$

### 4.3. Rupture criterion

Kachanov [8] assumed that the final rupture of the material in creep occurs when the scalar damage variable $D$ is equal to unity. In fact, there are critical values of the scalar damage variable corresponding to the rupture of the element [35,36]. Lemaitre suggested that for metals this value is less than one and varies from 0.2 to 0.8. For our three-dimensional model we assume that creep rupture occurs when the first principal value $D_I$ of the damage tensor is equal to a critical value $D_R$. The resulting material parameters of the present damage model are $\alpha_1, \alpha_2, (\alpha_3 = 1 - \alpha_1 - \alpha_2)$, $B, n, p, m, q$ and $D_R$.

## 5. Creep model coupled with material damage

Replacing the stress in Eqs. (1), (2), (7a), (7b), (7c) and (7d) by the effective stress defined in Eq. (22), we obtain the anisotropic creep model coupled with material damage

$$\dot{\boldsymbol{\varepsilon}} = \overset{\langle 4 \rangle}{\mathbf{A}}_1 : \dot{\tilde{\boldsymbol{\sigma}}} + \overset{\langle 4 \rangle}{\mathbf{A}}_2 : \dot{\tilde{\boldsymbol{\sigma}}} + \overset{\langle 4 \rangle}{\mathbf{A}}_3 : \boldsymbol{\tau}, \quad (39)$$

$$\dot{\boldsymbol{\tau}} = \overset{\langle 4 \rangle}{\mathbf{A}}_4 : \dot{\tilde{\boldsymbol{\sigma}}} + \overset{\langle 4 \rangle}{\mathbf{A}}_5 : (\tilde{\boldsymbol{\sigma}} - \boldsymbol{\tau}), \quad (40)$$

where the material tensors $\overset{\langle 4 \rangle}{\mathbf{A}}_i$ $(i = 1, \ldots, 4)$ depend on the cubic invariants of the effective stresses analogous to Eqs. (3a)–(8). Note that a transformation from $\boldsymbol{\tau}$ to some $\tilde{\boldsymbol{\tau}}$ as for $\boldsymbol{\sigma}$ is not needed.

## 6. Theoretical predictions and comparison with experimental results

Bertram and Olschewski [6] have identified the material parameters of their creep model for the nickel-based superalloy SRR99 at 760°C. The corresponding material parameters of the damage model are adopted from the work of Qi and Bertram [37], and are shown in Table 1. It is easy to see from the damage evolution law (28) and (36) that only the parameters $B$, $n$ and $p$ will be active if an uniaxial load acts in the [0 0 1]-orientation. Therefore we recommend to identify the parameters $B$, $n$ and $p$ at first by calibrating the numerical values by uniaxial experiments with specimens of [0 0 1]-orientation. For the identification of material parameters, the optimization packages MINPACK and NAG Libraries have been used. The Heun-method has been applied for the numerical integration.

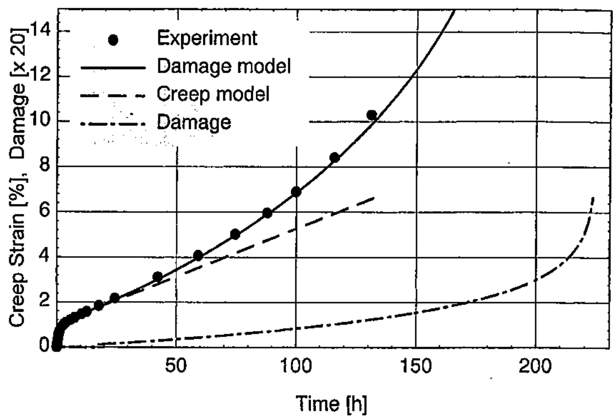


Fig. 1. Model prediction and experimental data of test MTU02 ($\varphi_1 = 8°$, $\varphi_2 = 8°$, $\sigma = 800$ MPa).

The creep tests were conducted at MTU and at BAM-V.21. Fig. 1 shows measured and simulated strains and the calculated damage of the creep test MTU02. The applied load is 800 MPa and the crystal orientation of the specimen is characterized by the Eulerian angles $\varphi_1 = 8°$ and $\varphi_2 = 8°$, determining the orientation of the crystal relative to the load axis. Note that $\varphi_2 = 0$ characterizes the ideal [0 0 1]-orientation. Creep model means the calculation by using Eqs. (1) and (2), and the *damage model* means the calculation by using Eqs. (39) and (40). *Damage* denotes the first principal value of the damage tensor **D** multiplied by 20 (so that it appears in the same range as the creep curve). Figs. 2 and 3 give the comparison of the theoretical predictions and the experimental data as well as the damage development for two tests with different orientations, MTU06 ($\varphi_1 = 45°$, $\varphi_2 = 15°$) and MTU21 ($\varphi_1 = 40°$, $\varphi_2 = 16°$), which are performed under the same loading conditions as

Table 1
Material parameters of damage model (SRR99 at 760°C)

| $\alpha_1$ | $\alpha_2$ | $B$ (MPa) | $n$ | $p$ | $m$ | $q$ | $D_R$ |
|---|---|---|---|---|---|---|---|
| 0.0 | 0.5 | 1442.0 | 14.13 | 0.4549 | 51.85 | −0.3133 | 0.2 |

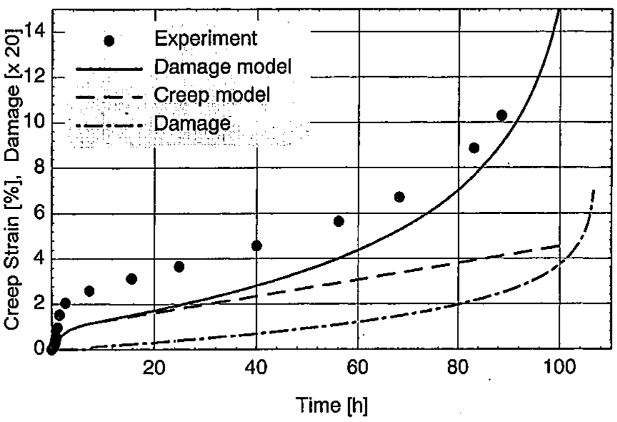


Fig. 2. Model prediction and experimental data of test MTU06 ($\varphi_1 = 45°$, $\varphi_2 = 15°$, $\sigma = 800$ MPa).

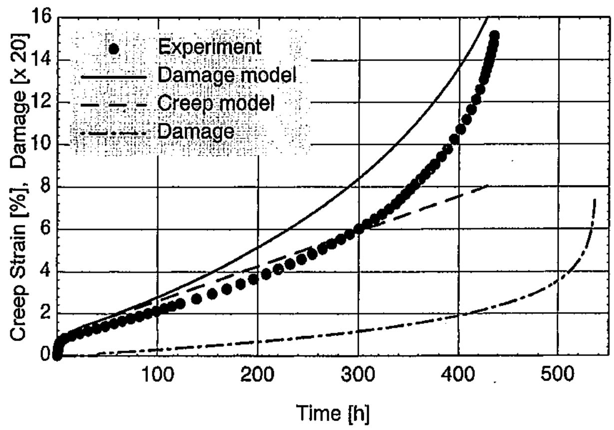


Fig. 4. Model prediction and experimental data of test BAM12 ($\varphi_1 = 0°$, $\varphi_2 = 8.5°$, $\sigma = 750$ MPa).

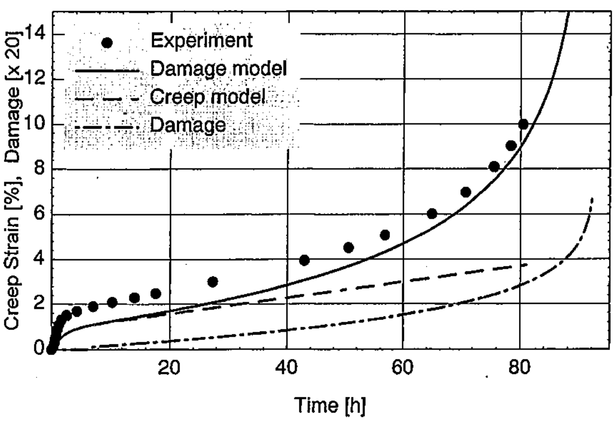


Fig. 3. Model prediction and experimental data of test MTU21 ($\varphi_1 = 40°$, $\varphi_2 = 16°$, $\sigma = 800$ MPa).

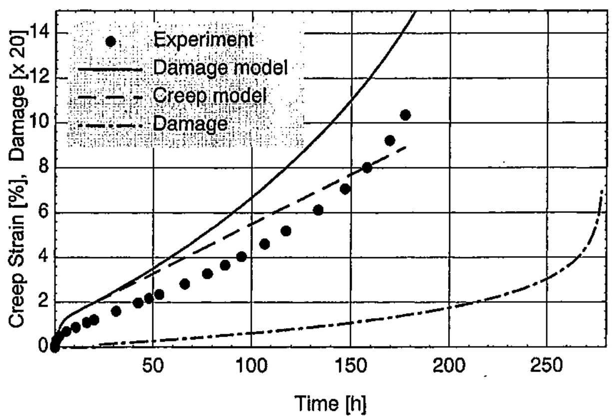


Fig. 5. Model prediction and experimental data of test MTU 20 ($\varphi_1 = 0°$, $\varphi_2 = 4°$, $\sigma = 800$ MPa).

MTU02. Note the different time-scales of the diagrams. Figs. 1–3 show that the used damage model reflects the anisotropic damage development and its influence on the anisotropic creep behavior. The coupled model with damage simulates the complete creep behavior till rupture, and can thus be used for the prediction of life time.

The strong nonlinearity of the stress–strain response is shown in Figs. 4–6 for a fixed crystal orientation. The difference between the model and the experimental data lies within the scatter band inherent to such tests.

The damage variable starts to grow rather early, so that the second or steady creep phase exists only in an approximate sense. This corresponds to the

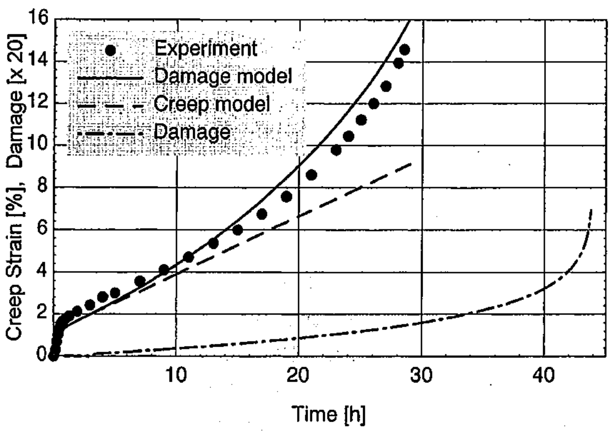


Fig. 6. Model prediction and experimental data of test BAM21 ($\varphi_1 = 0°$, $\varphi_2 = 7.5°$, $\sigma = 900$ MPa).

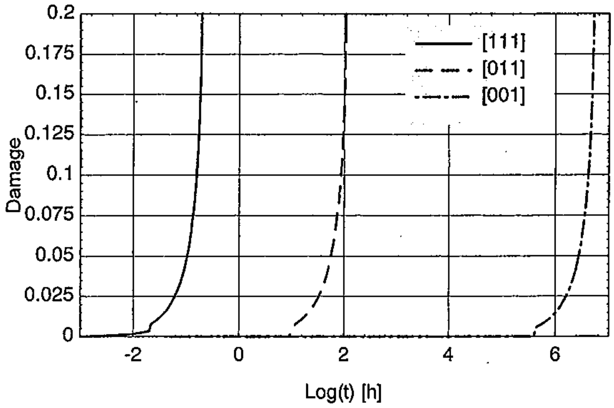


Fig. 7. Modeling of the damage development for the three extreme orientations ($\sigma = 400$ MPa).

experimental finding that the overlinearity of the creep curve of superalloys begins right after the state of minimal creep rate. Fig. 7 shows the theoretical prediction of the damage development under constant load $\sigma = 400$ MPa for the three extreme orientations.

## 7. Conclusion

A phenomenological model for the description of creep damage of single crystal superalloys is presented. Its application to the single crystal superalloy SRR99 demonstrates that the presented model provides satisfying prediction of the damage process and the life time in the range of the uniaxial monotonous creep. Both the influence of the material symmetry on the creep behavior and the nonlinearity of the material response with respect to the applied loads are reproduced by the present model. In order to test the capability of this model, especially for the case of multiaxial loading condition, further experimental investigations are needed. In this case, the choice of the relationship between the material parameters $\alpha_1$, $\alpha_2$ and $\alpha_3$ plays an important role. As it is rather difficult and costly to conduct multiaxial experiments for single crystals, theoretical investigations on the development of damage under multidimensional loads and in particular on its dependence on nonproportional loads are of special interest.

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