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An alternative approach to finite plasticity based on material isomorphisms

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Abstract

A framework for material models describing finite plastic deformations is established by the assumption of isomorphic elastic ranges. The concepts of decomposition into elastic and plastic deformations is not needed, neither intermediate configurations. A comparison with other approaches is given and shows their range of validity.

1. Introduction

In his Critical Review of the State of Plasticity, Naghdi (1990) drew the conclusion, that "there is some degree of disagreements on nearly all of the main constitutive ingredients and features of plasticity in the presence of finite deformation... Some of the issues of disagreements are of basic and fundamental importance." Today, almost one decade later, and after more than three decades of intensive research and publishing activities in this field, Naghdi's conclusion seems to be still valid. There are numerous constitutive models that intend to describe finite plastic deformations, and there are many FEM-codes in use with those models implemented for the computation of such problems. However, different "schools" are still competing, and a generally accepted framework for these theories is still lacking.

This fact is difficult to understand, for at least three reasons:

- There is great need for a general theory of finite plasticity, as there are many applications for it in metal-forming, granular materials technology, etc.

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- The physics of the micro-mechanics has been intensively studied and deeply understood. Both metal physicists and experimentalists are able to answer most of the questions that a constitutive lawyer could raise.
- Since the late fifties, the school of Rational Mechanics has tried hard to establish a general framework for the mechanical and thermodynamical behavior of *all* materials. The outcome was a satisfying theory for elastic and viscoelastic solids and viscous fluids, but no convincing plasticity theory was offered. Only few authors from this school considered plasticity, e.g. Owen (1968, 1970), Kratochvil (1973), Del Piero (1975), Wang (1975), Silhavy (1977), Silhavy and Kratochvil (1977), Krawietz (1986), Lucchesi and Podio-Guidugli (1988, 1990), Lucchesi et al. (1992). But very likely none of their suggestions was ever really accepted and widely used, mainly because they were too complicated and far from application. The contributions from Rational Mechanics in the field of finite plasticity were hardly influential in comparison to those on elasticity or viscoelasticity.

With the exceptions of few early suggestions such as Eckart (1948), Bilby et al. (1955), Kondo (1952), and Kröner (1960), the history of finite plasticity started in the late sixties with the historical works of the American and the French schools. Besides uncounted suggestions in the field that were forgotten already soon after their appearence, the mainstream of today's plasticity theory is threefold:

1. the concept of an *additive decomposition* of the strain tensor into an elastic and a plastic part, both assumed to be symmetric, starting from Green and Naghdi (1965);
2. the concept of an *unloaded intermediate placement* together with a *multiplicative decomposition* of the deformation gradient suggested by Lee (1969) and Mandel (1971) in the same period;
3. theories without elastic ranges as generalizations of integral equations from finite linear viscoelasticity, such as the endochronic theory (Valanis, 1971, Haupt, 1977).

While both the first and second approach try to generalize the concepts of classical Prandtl–Reuss–v. Mises plasticity theory and are thus comparable and competitive, the latter is different in its fundamental concepts and phenomena and will therefore not be considered in the present context. We will instead limit our considerations to materials with elastic ranges as it is substantiated, e.g. by crystal plasticity. In the case of crystal plasticity, some of the basic mechanisms are much better understood than in phenomenological plasticity.[1] It should be therefore possible to consider crystal plasticity as a specialization of general plasticity.

For our purpose we will construct a framework for constitutive models that show the characteristic features of elasto-plasticity as we know them from classical plasticity.

[1]See, e.g. Asaro (1983), Havner (1992), Hosford (1993), Khan and Huang (1995), Khan and Cheng (1996), Kratochvil and Dillon (1970), Kratochvil (1971), Krawietz (1986).

It is not meant that these models are capable to describe each and every effect that can be observed in metals or other plastic materials, but instead characterizes some basic properties of plastic behavior. We consider these properties as being so typical that we expect every plasticity theory to be capable to include them. As such we choose:

- the existence of elastic ranges at any instant for the material point being limited by a yield limit such that the material point behaves like an elastic point as long as the further deformation does not exceed this limit.
- the elastic isomorphy of the elastic behavior within the elastic ranges.

We know from crystal plasticity that the current deformation of the lattice and not of the material determines the stresses. But as plastic deformations do not change the mean properties of the lattice, the characteristic parameters that determine the elastic properties, shall not be altered by plastic deformations (Anand and Brown, 1987 p. 9)[2]

- rate-independence of plasticity.

Although rate-dependent effects can be observed in most, if not all materials, for typical plastic behavior these are commonly neglected. In the present context we will restrict ourselves to rate-independent behavior in order to keep the structure of the models as simple as possible without questioning the importance of viscoplasticity.

While the above assumptions restrict our considerations to a special type of behavior, namely elasto-plastic behavior, we require full generality with respect to the following aspects:

- three-dimensionality of the model;
- large deformations, which requires material frame-indifference of all constitutive equations involved;
- anisotropy of any type of all constitutive aspects;
- thermodynamic consistency, i.e. satisfaction of the second law of thermodynamics;
- hardening and softening behavior, or materially stable and unstable behavior;
- associated and non-associated flow rules.

One of the shortcomings of finite plasticity theories that Naghdi (1990) mentioned, is the confusion in the choice of dependent and independent variables and their distribution (or decomposition) into elastic and plastic parts. In the mechanical theory to follow, deformation-processes are generally considered as the independent variables, which can be freely prescribed by the designer of experiments. The response of the material is expressed by stresses and will be considered as the dependent variable. In between these two groups of variables, a set of internal variables will be

[2]*Note added in proof*: In forthcoming papers by Rajagopal and Srinivasa quite similar assumptions have been suggested. Their concept of *natural configurations* is not far from our use of material isomorphisms. The author is grateful to Arun Srinivasa for this helpful information.

introduced as the handling of such variables (real numbers) is more practical than that of processes (functions). The internal variables do not explicitly appear in the balances and, thus, are not (directly) observable.

This choice of variables is influenced by Rational Mechanics where it is taught that the process of some appropriately chosen deformation tensor determines the stresses at the end of the process. And the appropriate choice of these strain and stress tensors is reduced by the principle of material frame-indifference. By choosing the invariant or material right Cauchy–Green-tensor and a work-conjugate (material) stress tensor, the entire theory indentically fulfills these invariance requirements.

As Noll (1972) pointed out, the concept of (semi-infinite) deformation histories can be used for materials with fading memory (semielasticity), but is generally inappropriate for rate-independent materials with permanent memory such as plastic ones (Bertram, 1993). Instead, we use deformation processes of finite duration starting from some (arbitrarily) fixed initial state. For the sake of simplicity and clearness, we will not overload this text and avoid the formalization or axiomatization of material systems (Noll, 1972, Bertram, 1982, 1989) but, instead, leave such notions as the *state* concept intuitive.

Based on the above assumptions we will establish an elasto-plasticity theory that does not contain any constitutive assumptions on the decomposition of deformations or rates into elastic or plastic parts. Instead, we will afterwards be able to derive different kinds of such decompositions.

Within some theories on plasticity, the concept of reference placement plays a crucial role although its physical interpretation in many cases is not clear. This has led to some confusion, especially when mixed up with changes of observer or superimposed rigid body motions. In 1972, Noll suggested the intrinsic description of strains and stresses which does not make use of any reference placement. Although the intrinsic description has great advantages and seems to be more natural than any other one, it did not become very popular and was used only by very few authors (Del Piero, 1975, Silhavy and Kratochvil, 1977, Krawietz, 1986, Bertram, 1989). The reason for this is perhaps that it needs some rather abstract geometrical concepts such as tangent and cotangent spaces on manifolds which are apparently not commonly available. Taking this situation into account, we decided to not make use of intrinsic description as in Bertram (1992), but instead refer to a reference placement. However, its choice will play no role, as it is absolutely arbitrary. And the variables to be used are rather similar to the intrinsic variables of Noll (1972), such that a transformation of the one set into the other is straight-forward.

2. Materials with isomorphic elastic ranges

We denote by

$\mathscr{R}$ the set of all real numbers

$\mathscr{V}$ the set of all vectors associated with the 3-dimensional Euclidean point space

$\mathscr{Lin}$	the set of all tensors on $\mathscr{V}$
$\mathscr{Inv}$	the set of all invertible tensors
$\mathscr{Sym}$	the set of all symmetric tensors
$\mathscr{Psym}$	the set of all symmetric and positive-definite tensors
$\mathscr{Orth}$	the set of all orthogonal tensors

A superscript $+$ indicates the restriction to tensors with positive determinant. $\otimes$ denotes the tensor product.

Let $\mathbf{F} \in \mathscr{Inv}$ be the deformation gradient and $\mathbf{T} \in \mathscr{Sym}$ the Cauchy stress tensor, then $\mathbf{C} := \mathbf{F}^T\mathbf{F} \in \mathscr{Psym}$ is the right Cauchy–Green configuration tensor and

$$\mathbf{S} := \mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T} \in \mathscr{Sym} \tag{1}$$

the material stress tensor,[3] being work-conjugate to $\mathbf{C}$, as the specific stress power is

$$\rho^{-1}\mathrm{tr}(\mathbf{TL}) = \rho^{-1} \quad {}^1\!/_2\,\mathrm{tr}(\mathbf{SC}^{\bullet}) \tag{2}$$

with the current mass density ρ. If the configuration $\mathbf{C}$ depends on time in some closed interval $[0,d]$, we call it a *configuration process* or a $\mathbf{C}$-*process*. We will always assume that such $\mathbf{C}$-processes are continuous and piecewise continuously differentiable. If we restrict a $\mathbf{C}$-process to a shorter interval $[0,\underline{d}]$, $\underline{d} < d$, we call it a *subprocess*. Conversely, the original process restricted to $[\underline{d},d]$ will be called *continuation* of the subprocess. Clearly, for compatibility a process can only continue some other process, if the first ends at the same time and with the same value when and where the second one starts. The independent quantities of our format are such $\mathbf{C}$-processes that start at some initial time 0 at the same initial configuration $\mathbf{C}(0)$. In Bertram (1982), such a set is called *process-class*, and for some material point a *material functional* assigns to each process out of this class the stress at its end. To specify such a general functional for elasto-plastic behavior will be our task next.

The constitutive ingredient for elasto-plasticity is the notion of elastic ranges. Let p be an index to be specified later.

Definition. An elastic range consists of $\{\mathscr{E}_p, h_p\}$ with

- $\mathscr{E}_p \subset \mathscr{Psym}$ being a path-connected closed subset of the configuration space that forms a differentiable manifold with boundary;[4]
- an elastic law

  $h_p : \mathscr{E}_p \to \mathscr{Sym} \mid \mathbf{C} \mapsto \mathbf{S}$,

  being continuously differentiable and as such extendible onto $\mathscr{Psym}$.

[3] By definition, this stress tensor is similar, but not equal to both the 2. Piola–Kirchhoff tensor and to Noll's intrinsic stress tensor. To our best knowledge, this tensor has not been used in literature, although it has nice properties, especially in the context of plasticity.

[4] For certain yield criteria such as Tresca's this assumption is too restrictive and should be weakend by assuming smoothness of the boundary *almost* everywhere. However, for the sake of simplicity we will exclude this possibility for the moment.

If the elastic law h_p allows for a potential w_p

$$\mathbf{S} = 2\rho \operatorname{grad} w_p(\mathbf{C}) \tag{3}$$

we will call $\{\mathscr{E}_p, h_p\}$ a *hyperelastic range*. The first assumption states the existence of elastic ranges for any **C**-process.

Assumption 1. For any **C**-process out of the process class there exists an elastic range $\{\mathscr{E}_p, h_p\}$ such that

- its final configuration is in $\mathscr{E}_p$, and
- for any continuation of this process that remains entirely in $\mathscr{E}_p$, the stresses at its end are determined by the elastic law h_p through its final configuration **C**

$$\mathbf{S} = h_p(\mathbf{C}) \tag{4}$$

Clearly, there are subsets of some elastic range with the same properties. To remove this ambiguity, we will always think of the *maximal* elastic range (without further mention).

In order to describe such a subset $\mathscr{E}_p$ of $\mathscr{Sym}$, it is quite practical to use the notion of an *yield criterion* associated with $\mathscr{E}_p$ which is just an indicator function

$$\mathbf{\Phi}_p : \mathscr{Sym} \to \mathscr{R} \mid \mathbf{C} \mapsto \mathbf{\Phi}_p(\mathbf{C})$$

such that

$$\begin{aligned} \mathbf{\Phi}_p(\mathbf{C}) < 0 &\iff \mathbf{C} \in \mathscr{E}_p^o \text{ (interior of } \mathscr{E}_p) \\ \mathbf{\Phi}_p(\mathbf{C}) = 0 &\iff \mathbf{C} \in \partial\mathscr{E}_p \text{ (boundary of } \mathscr{E}_p, \text{called } \textit{yield limit}) \\ \mathbf{\Phi}_p(\mathbf{C}) > 0 &\iff \mathbf{C} \in \mathscr{Sym}\backslash\mathscr{E}_p \text{(elsewhere)} \end{aligned}$$

Such a yield criterion trivially exists for any elastic range, but it is by no means unique. We further assume that $\mathbf{\Phi}_p$ is a differentiable function on $\mathscr{Sym}$. By the chain rule,

$$\mathbf{\Phi}_p{}^{\bullet} = \operatorname{tr}\{\operatorname{grad} \mathbf{\Phi}_p(\mathbf{C})\mathbf{C}^{\bullet}(t)\} \tag{5}$$

holds in all cases.

As an example, we consider the Huber–v. Mises J_2-criterion. Let $J_2(\operatorname{dev}\mathbf{T})$ be the second invariant of the deviatoric Cauchy stress dev **T**, and σ_{Fp} the yield limit under uniaxial tension. Then $\{-3J_2(\operatorname{dev}\mathbf{T})\}^{1/2}$ is the usual v. Mises equivalent stress. It is easy to see that **T** and **SC** have the same principal invariants. Hence,

$$J_2(\operatorname{dev}\mathbf{T}) = J_2(\operatorname{dev}\mathbf{SC}) \tag{6}$$

and

$$\Phi_p(\mathbf{C}) = \{-3J_2(\mathrm{dev}\, h_p(\mathbf{C})\mathbf{C})\}^{1/2} - \sigma_{Fp} \tag{7}$$

is an associated yield criterion (without hardening) for the Huber–v. Mises yield limit.

If the **C**-process is such that the current **C** lies in the interior $\mathscr{E}_p^o$ of $\mathscr{E}_p$, the material behaves elastically and the current state will be called *elastic*. If the configuration lies on the yield limit $\partial\mathscr{E}_p$, i.e. $\Phi_p = 0$, two cases are possible.

i. The **C**-process does not exceed the yield limit because unloading

$$\Phi_p^{\bullet} < 0 \tag{8}$$

or neutral loading

$$\Phi_p^{\bullet} = 0 \tag{9}$$

occurs. That means that the **C**-process is directed towards the interior of $\mathscr{E}_p$, or is tangential to the yield surface, respectively. In both cases, the state is again *elastic*.

ii. Only if both the *yield condition*

$$\Phi_p = 0 \tag{10}$$

and the *loading condition*

$$\Phi_p^{\bullet} > 0 \tag{11}$$

are fulfilled simultaneously, the process tends to leave the current elastic range and, thus, changes it. Such a state will be called *plastic* or *yield state*. If a state-process passes only through elastic (plastic) states, we will call it *elastic (plastic) process*.

It should be emphasized that the loading condition does not exclude softening behavior, as the elastic ranges are defined in the strain space and not in the stress space (see Naghdi and Trapp, 1975, Casey and Naghdi, 1981).

During plastic processes, the material is continuously changing its current elastic ranges. This means that not only the set $\mathscr{E}_\mathrm{p}$ is changing and, as a consequence, the function Φ_p does, but also that the elastic law h_p has to vary. All functions and variables with suffix p vary (only) during plastic processes, but remain constant in elastic ones.

For most metals, however, the change of the elastic constants due to yielding is negligible. To express this idea in precise mathematical terms, the concept of (elastic) *material isomorphisms* is needed as we know it from elasticity theory (see Noll, 1972; Wang and Truesdell, 1973; Bertram, 1982, 1989). This is laid down in the following.

Assumption 2. Let $\{\mathscr{E}_1, h_1\}$ and $\{\mathscr{E}_2, h_2\}$ be two elastic ranges of the same material point. Then h_1 and h_2 are isomorphic, i.e. there exists a material isomorphism $\mathbf{P}_{12} \in \mathscr{I}_{\mathrm{m}}$ such that

$$h_2(\mathbf{C}) = \mathbf{P}_{12} h_1(\mathbf{P}_{12}^T \mathbf{C} \mathbf{P}_{12}) \mathbf{P}_{12}^T \tag{12}$$

holds for all $\mathbf{C} \in \mathscr{P}_{\mathrm{sym}}$.

As we extrapolated the elastic laws onto the entire space $\mathscr{P}_{\mathrm{sym}}$, there is no need to distinguish between the original domains $\mathscr{E}_1$ and $\mathscr{E}_2$ of the two elastic functions in this assumption.

By means of this assumption, we can choose one (arbitrary) elastic range, say $\{\mathscr{E}_0, h_0\}$, as a (constant) *elastic reference range*, and then transform all other elastic laws h_p by some $\mathbf{P} \in \mathscr{I}_{\mathrm{m}}$ into h_0

$$\mathbf{S} = h_p(\mathbf{C}) = \mathbf{P} h_0(\mathbf{P}^T \mathbf{C} \mathbf{P}) \mathbf{P}^T \tag{13}$$

While h_p is continuously changing as a function during plastic processes, the right hand side does not change as a function, but only through the time-dependent variables $\mathbf{P}$ and $\mathbf{C}$. We will call (13) the *isomorphy condition* and $\mathbf{P}$ the *plastic transformation* (see Wang and Bloom, 1974). $\mathbf{P}$ will not be interpreted as a deformation tensor. It has to be considered as an internal variable for which an evolution equations is needed. Before we deal with this, however, the question of the uniqueness of $\mathbf{P}$ arises. As is shown in the next theorem, this question is directly connected with the symmetry properties of the elastic laws. We recall that a tensor $\mathbf{A} \in \mathscr{I}_{\mathrm{m}}$ is a *symmetry transformation* (material automorphism) of an elastic law h if

$$h(\mathbf{C}) = \mathbf{A}\, h(\mathbf{A}^T \mathbf{C} \mathbf{A}) \mathbf{A}^T \tag{14}$$

is satisfied for all $\mathbf{C} \in \mathscr{P}_{\mathrm{sym}}$. All such symmetry transformations form the *symmetry group* $\mathscr{G}$ of h, being characteristic for each material.

Theorem. *Let $\{\mathscr{E}_0, h_0\}$ and $\{\mathscr{E}_p, h_p\}$ be two elastic ranges of an elasto-plastic material. Then the following facts hold.*

1. *(Noll, 1958) If $\mathbf{P}$ is a plastic transformation from $\mathscr{E}_0$ to $\mathscr{E}_{\mathrm{p}}$ and $\mathscr{G}_0$ the symmetry group of h_0, then*

$$\mathscr{G}_p = \mathbf{P} \mathscr{G}_0 \mathbf{P}^{-1} \tag{15}$$

   is the symmetry group of h_p.
2. *If $\mathbf{P}$ is a plastic transformation from $\mathscr{E}_0$ to $\mathscr{E}_p$ and $\mathscr{G}_o$ and $\mathscr{G}_p$ the symmetry groups of h_0 and h_p, respectively, then $\mathbf{A}_p \mathbf{P} \mathbf{A}_0$ too is a plastic transformation from $\mathscr{E}_0$ to $\mathscr{E}_p$ for all $\boldsymbol{A}_0 \in \mathscr{G}_0$ and $\boldsymbol{A}_p \in \mathscr{G}_p$.*
3. *Let $\mathbf{P}$ and $\underline{\mathbf{P}}$ be plastic transformations from $\mathscr{E}_0$ to $\mathscr{E}_p$, then*

$$\mathbf{P}\underline{\mathbf{P}}^{-1} \in \mathscr{G}_p \quad \text{and} \quad \mathbf{P}^{-1}\underline{\mathbf{P}} \in \mathscr{G}_0.$$

By the above definitions, the proof of this theorem is straightforward and omitted for brevity (see Bertram, 1992).

If we apply the second part of the theorem to our format, we can conclude that *the plastic transformation between two elastic ranges is unique only up to both-sided symmetry transformations*. An important consequence of this fact is obtained if the reference placement of h_0 is an isotropic state, i.e. the symmetry group of h_0 is the general orthogonal group $\mathcal{Orth}$. If $\mathbf{P}$ is the plastic transformation to some other elastic range, then so is $\mathbf{PQ}$ for all orthogonal tensors $\mathbf{Q}$. By an appropriate choice of $\mathbf{Q}$, we can always generate a symmetric plastic transformation in this case. Thus, *if the elastic reference law is isotropic, the plastic transformation can be taken as symmetric. For any kind of anisotropy*, however, this is not the case, and $\mathbf{P}$ *must be considered as non-symmetric in general*.

By the isomorphy condition, we are able to reduce the time-dependence of the function h_p during yielding to the time-dependence of the tensorial variable $\mathbf{P}$. In order to do the same for the evolution of the set $\mathscr{E}_p$ during yielding, or, equivalently, for the associated yield criterion $\mathbf{\Phi}_p$, we assume that there exists a (hardening) variable $\mathbf{Z}_p$ out of some linear space of finite dimension $\underline{\mathscr{Lin}}$, and a function

$$\varphi : \mathscr{Inv} \times \mathscr{Sym} \times \underline{\mathscr{Lin}} \to \mathscr{R}$$

such that

$$\mathbf{\Phi}_p(\mathbf{C}) = \varphi(\mathbf{P}, \mathbf{C}, \mathbf{Z}_p) \tag{16}$$

holds for all elastic ranges. φ depends on the current elastic range only through its arguments $\mathbf{P}$ and $\mathbf{Z}_p$ being internal variables. Of course, the yield condition and the loading condition can be expressed by φ through

$$\mathbf{\Phi}_p(\mathbf{C}) = \varphi(\mathbf{P}, \mathbf{C}, \mathbf{Z}_p) = 0 \tag{17}$$

and

$$\mathbf{\Phi}_p(\mathbf{C})^{\bullet} = \mathrm{tr}\{\frac{\mathrm{d}\mathbf{\Phi}_p(\mathbf{C})}{\mathrm{d}\mathbf{C}}\mathbf{C}^{\bullet}\} = \mathrm{tr}\{\frac{\partial\varphi}{\partial\mathbf{C}}\mathbf{C}^{\bullet}\} > 0. \tag{18}$$

As an example, we again consider the Huber–v.Mises-yield criterion, but with some back-stress $\mathbf{S}_{Bp}$ for the kinematic hardening and some isotropic hardening variable σ_{Fp}. In this case

$$\mathbf{\Phi}_p(\mathbf{C}) = \{-3J_2(\mathrm{dev}[h_p(\mathbf{C}) - \mathbf{S}_{Bp}]\mathbf{C})\}^{1/2} - \sigma_{Fp} \tag{19}$$

or

$$\varphi(\mathbf{P}, \mathbf{C}, \mathbf{Z}_p) = \{-3J_2(\mathrm{dev}[\mathbf{P}h_0(\mathbf{P}^T\mathbf{C}\mathbf{P})\mathbf{P}^T - \mathbf{S}_{Bp}]\mathbf{C})\}^{1/2} - \sigma_{Fp} \tag{20}$$

Here we identify $\mathbf{Z}_p \equiv \sigma_{Fp}, \mathbf{S}_{Bp} \in \underline{\mathscr{L}_{in}} \equiv \mathscr{R} \times \mathscr{S}_{ym}$.

For the internal variables $\mathbf{P}$ and $\mathbf{Z}_p$ we need evolution equations.

Assumption 3. There exist two rate-independent[5] evolution equations

$$p : \mathscr{I}_{nv} \times \mathscr{B}_{ym} \times \underline{\mathscr{L}_{in}} \times \mathscr{S}_{ym} \to \mathscr{L}_{in}$$

$$z : \mathscr{I}_{nv} \times \mathscr{B}_{ym} \times \underline{\mathscr{L}_{in}} \times \mathscr{S}_{ym} \to \underline{\mathscr{L}_{in}},$$

with

$$\mathbf{P}^{\bullet}\mathbf{P}^{-1} = p(\mathbf{P}, \mathbf{C}, \mathbf{Z}_p, \mathbf{C}^{\bullet}) \tag{21}$$

and

$$\mathbf{Z}_p^{\bullet} = z(\mathbf{P}, \mathbf{Z}_p, \mathbf{C}^{\bullet}) \tag{22}$$

so that

$$\mathbf{\Phi}_p(\mathbf{C}) = \varphi(\mathbf{P}, \mathbf{C}, \mathbf{Z}_p) \tag{23}$$

holds for the yield criteria associated with all elastic ranges.

We will refer to p and z as the *flow rule* and the *hardening rule*, respectively.

The flow rule posesses a *plastic potential*

$$\pi : \mathscr{L}_{in} \to \mathscr{R} \quad | \quad \mathbf{SC} \mapsto \pi(\mathbf{SC})$$

if $\mathbf{P}^{\bullet}\mathbf{P}^{-1}$ is parallel to the gradient of π

$$\mathbf{P}^{\bullet}\mathbf{P}^{-1} \parallel \operatorname{grad}^T \pi(\mathbf{SC}).$$

The flow rule is *associated* to the yield criterion, if the plastic potential equals the equivalent stress in the yield criterion, e.g.

$$\pi(\mathbf{SC}) \equiv \{-3J_2(\operatorname{dev}\mathbf{SC})\}^{1/2} \tag{24}$$

in the Huber–v. Mises case. As a consequence, $\mathbf{P}^{\bullet}\mathbf{P}^{-1}$ is in the direction of the deviatoric part of $\mathbf{SC}$. For generality, however, we will not restrict ourselves to these special cases in what follows.

The three material functions p, z, and φ cannot be choosen independently, but are linked together by the consistency requirement, as we will see next.

[5]In the context of viscoplasticity, these two functions would be rate-dependent. This possibility, however, has been excluded from the present paper for the sake of simplicity.

During elastic processes, all internal variables must remain constant. Thus, p and z contain a switcher that switches their values to zero whenever the yield condition or the loading condition are not fulfilled. In what follows, we consider exclusively plastic or yield states, i.e. we assume that Eqs. (17) and (18) are simultaneously fulfilled (which implies $\mathbf{C}(t)^{\bullet} \neq \mathbf{0}$).

Because of the rate-independence, p and z are positive homogeneous of degree one in their last argument $\mathbf{C}^{\bullet}$. Hence

$$p(\mathbf{P}, \mathbf{C}, \mathbf{Z}_p, \lambda\mathbf{C}^{\bullet}) = \lambda\, p(\mathbf{P}, \mathbf{C}, \mathbf{Z}_p, \mathbf{C}^{\bullet}) \tag{25}$$

and

$$z(\mathbf{P}, \mathbf{C}, \mathbf{Z}_p, \lambda\mathbf{C}^{\bullet}) = \lambda\, z(\mathbf{P}, \mathbf{C}, \mathbf{Z}_p, \mathbf{C}^{\bullet}) \tag{26}$$

hold for all positive numbers λ, and especially for $\lambda := |\mathbf{C}^{\bullet}|$. In this case,

$$\mathbf{P}^{\bullet}\mathbf{P}^{-1} = \lambda\, \Pi(\mathbf{P}, \mathbf{C}, \mathbf{Z}_p, \mathbf{C}^{\circ}) \tag{27}$$

and

$$\mathbf{Z}^{\bullet} = \lambda\, Z(\mathbf{P}, \mathbf{C}, \mathbf{Z}_p, \mathbf{C}^{\circ}) \tag{28}$$

with $\mathbf{C}^{\circ} := \mathbf{C}^{\bullet}/\lambda$, where Π and Z are the restrictions of p and z to tensors of norm 1 in the last argument

$$\Pi(\mathbf{P}, \mathbf{C}, \mathbf{Z}_p, \mathbf{C}^{\circ}) := p(\mathbf{P}, \mathbf{C}, \mathbf{Z}_p, \mathbf{C}^{\circ}) \tag{29}$$

and

$$Z(\mathbf{P}, \mathbf{C}, \mathbf{Z}_p, \mathbf{C}^{\circ}) := z(\mathbf{P}, \mathbf{C}, \mathbf{Z}_p\mathbf{C}^{\circ}). \tag{30}$$

During yielding, the material must permanently remain on the current yield limit

$$\begin{aligned} 0 &= \varphi(\mathbf{P}, \mathbf{C}, \mathbf{Z}_p)^{\bullet} \\ &= \operatorname{tr}\left\{\left(\frac{\partial\varphi}{\partial\mathbf{P}}\right)^{T}\mathbf{P}^{\bullet}\right\} + \operatorname{tr}\left\{\frac{\partial\varphi}{\partial\mathbf{C}}\mathbf{C}^{\bullet}\right\} + \operatorname{tr}\left\{\left(\frac{\partial\varphi}{\partial\mathbf{Z}_p}\right)^{T}\mathbf{Z}_p{}^{\bullet}\right\} \\ &= \operatorname{tr}\left\{\left(\frac{\partial\varphi}{\partial\mathbf{P}}\right)^{T}\lambda\Pi\mathbf{P}\right\} + \operatorname{tr}\left\{\frac{\partial\varphi}{\partial\mathbf{C}}\mathbf{C}^{\bullet}\right\} + \operatorname{tr}\left\{\left(\frac{\partial\varphi}{\partial\mathbf{Z}_p}\right)^{T}\lambda Z\right\}. \end{aligned} \tag{31}$$

This equation determines λ, and we obtain the *consistent flow rule*

$$\mathbf{P}^{\bullet}\mathbf{P}^{-1} = \Pi \otimes \mathbf{A}[\mathbf{C}^{\bullet}] \tag{32}$$

and the *consistent hardening rule*

$$\mathbf{Z}^{\bullet} = Z \otimes \mathbf{A}[\mathbf{C}^{\bullet}] \tag{33}$$

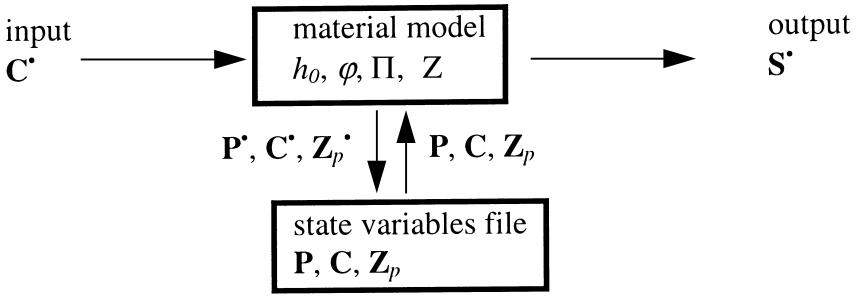
with

$$\mathbf{A}(\mathbf{P}, \mathbf{C}, \mathbf{Z}_p, \mathbf{C}^{\circ}) := -\mathrm{tr}\left[\Pi\mathbf{P}\left(\frac{\partial\varphi}{\partial\mathbf{P}}\right)^{T} + Z\left(\frac{\partial\varphi}{\partial\mathbf{Z}_p}\right)^{T}\right]^{-1}\frac{\partial\varphi}{\partial\mathbf{C}}. \tag{34}$$

In conclusion, within this frame work the elasto-plastic material element is determined by the following material functions

$h_0(\mathbf{C})$
$\varphi(\mathbf{P}, \mathbf{C}, \mathbf{Z}_p)$
$\Pi(\mathbf{P}, \mathbf{C}, \mathbf{Z}_p, \mathbf{C}^{\circ})$
$Z(\mathbf{P}, \mathbf{C}, \mathbf{Z}_p, \mathbf{C}^{\circ})$

and initial values for $\mathbf{P}$ and $\mathbf{Z}_p$. The deformation process can be arbitrarily prescribed. Then by means of the above constitutive equations, we are able to determine the accompanying $\mathbf{P}$- and $\mathbf{Z}_p$- and $\mathbf{S}$-process as a result of integration of the evolution functions. For numerical purposes, the material model can be considered as the following scheme.



By the chain rule we can find a piecewise linear incremental form for the stresses

$$\mathbf{S}^{\bullet} = \mathbf{K}_p[\mathbf{C}^{\bullet}] \tag{35}$$

where the elastoplastic tangent operator $\mathbf{K}_p$ depends not only on the current state variables $\mathbf{P}$, $\mathbf{C}$, $\mathbf{Z}_p$, but also on the direction of $\mathbf{C}^{\bullet}$ that determines whether the state is elastic or plastic. In the former case, $\mathbf{K}_p$ coincides with the current elastic stiffness tensor grad h_p. In the latter case, however, $\mathbf{K}_p$ depends also on the hardening rule.

Alternatively to this incremental form for $\mathbf{S}^{\bullet}$, one can also use the finite form (13) for determining the stresses.

The stress-power per unit mass (2) can be split into two parts

$$\begin{aligned}
&\rho^{-1}\ {}^1\!/_2\mathrm{tr}(h_p(\mathbf{C})\mathbf{C}^\bullet)\\
&=\rho^{-1}\ {}^1\!/_2\mathrm{tr}\{\mathbf{P}h_0(\mathbf{P}^T\mathbf{C}\mathbf{P})\mathbf{P}^T\mathbf{C}^\bullet\}\\
&=\rho^{-1}\ {}^1\!/_2\mathrm{tr}\{h_0(\mathbf{C}_0)\mathbf{C}_0^\bullet\}-\rho^{-1}\ \mathrm{tr}\{h_0(\mathbf{C}_0)\mathbf{P}^T\mathbf{C}\mathbf{P}^\bullet\mathbf{P}^{-1}\mathbf{P}\}\\
&=\rho^{-1}\ {}^1\!/_2\mathrm{tr}\{h_0(\mathbf{C}_0)\mathbf{C}_0^\bullet\}-\rho^{-1}\ \mathrm{tr}\{\mathbf{S}\mathbf{C}\Pi\otimes\mathbf{A}[\mathbf{C}^\bullet]\mathbf{P}\}
\end{aligned}\tag{36}$$

with $\mathbf{C}_0:=\mathbf{P}^T\mathbf{C}\mathbf{P}$. If the elastic range is also hyperelastic, the first term is just the change $w_0^\bullet$ of the elastic reference energy. According to the second law of thermodynamics, the second term is dissipative, i.e. never negative (Acharya and Shawki 1996, Bertram 1999).

3. Crystal plasticity

Since the early nineteen-twenties, we know that plastic deformations in crystals at room temperature take place as shears in crystallographic slip-systems. The deformation of the lattice determines the stresses, such that a distinction between material and lattice deformations is natural for crystals. However, we will not go as far as Krawietz (1986) and Rubin (1996) who consider the lattice space as a primitive concept. In our context, the notion of a lattice is just an illustrative means for the action of the plastic transformation, as we will see later.

A slip system is a pair $\{\mathbf{d}_\alpha,\mathbf{n}^\alpha\}$ of a director $\mathbf{d}_\alpha\in\mathscr{V}$ and a covector $\mathbf{n}^\alpha\in\mathscr{V}^*$ which indicate the direction and the normal of the plane, respectively, in which the crystallographic slip occurs. Here, $\mathscr{V}^*$ stands for the dual space of $\mathscr{V}$, and α is an index for the particular slip system. As before, we prefer a material description. Thus, $\mathbf{d}_\alpha$ and $\mathbf{n}^\alpha$ indicate crystallographic directions in the (arbitrarily chosen) reference placement.

A slip-system remains inactive, as long as the *resolved shear stress*

$$\tau_s^\alpha:=\mathrm{tr}(\mathbf{S}\mathbf{C}\mathbf{d}_\alpha\otimes\mathbf{n}^\alpha)\tag{37}$$

remains (in absolute value) below the critical resolved shear stress τ_c^α of that particular slip system. The Schmid-law is the associated yield criterion

$$\begin{aligned}
\Phi_p(\mathbf{C})&=\max_\alpha|\tau_s^\alpha|-\tau_c^\alpha=\max_\alpha|\mathrm{tr}(\mathbf{S}\mathbf{C}\mathbf{d}_\alpha\otimes\mathbf{n}^\alpha)|-\tau_c^\alpha\\
&=\max_\alpha|\mathrm{tr}(\mathbf{P}h_0(\mathbf{P}^T\mathbf{C}\mathbf{P})\mathbf{P}^T\mathbf{C}\mathbf{d}_\alpha\otimes\mathbf{n}^\alpha)|-\tau_c^\alpha.
\end{aligned}\tag{38}$$

If only one slip system is activated (single slip), the Schmid criterion serves as a plastic potential and the associated flow rule is

$$\mathbf{P}^\bullet\mathbf{P}^{-1}=-\gamma^{\cdot\alpha}\,\mathrm{grad}^T\quad\tau_s(\mathbf{S}\mathbf{C})=-\gamma^{\cdot\alpha}\mathbf{d}_\alpha\otimes\mathbf{n}^\alpha\tag{39}$$

with a real number $\gamma^{\cdot\alpha}$. In the case of multiple slip, we sum the right hand side over all active slip systems. For each of them, a scalar has to be determined by the consistency

condition. It is well known that this is only possible if not more than five slip systems are activated. If this is not the case, an additional selection criterion has to be established that selects maximal five active slip systems out of the set of those slip systems where the critical resolved yield stress is reached (Anand and Kothari, 1996, Miehe and Schröder, 1998).

In the simplest case, isotropic hardening is taken into account by a linear ansatz

$$\tau_c^{\alpha\cdot} = \sum_\beta \Gamma_{\alpha\beta}\, \dot{\gamma}^\beta, \tag{40}$$

where $\Gamma_{\alpha\beta}$ describes self-hardening for $\alpha = \beta$ and latent or cross-hardening for $\alpha \neq \beta$. The vector of the internal variables can therefore be identified by $\mathbf{Z}_p \equiv \{\tau_c^1, \tau_c^2, \ldots, \tau_c^n\}$ where n is the total number of slip systems, depending on the specific crystal class under consideration.

In the case of rate-independent crystal plasticity, the functional forms for p and z can hardly be given explicitly. The scheme of the constitutive equations is rather coupled and implicit. Essentially, however, the state variables $\mathbf{P}$, $\mathbf{C}$, $\mathbf{Z}_p$ and the increment $\mathbf{C}^\bullet$ again determine all increments $\mathbf{P}^\bullet$, $\mathbf{Z}_p{}^\bullet$, $\mathbf{S}^\bullet$.

In most crystalline materials, the elastic range is rather small so that a linear elastic law is justified

$$h_p(\mathbf{C}) = \mathbf{K}_p[\mathbf{C} - \mathbf{C}_{\mathrm{u}p}] \tag{41}$$

with the material constants

$\mathbf{K}_p$: the fourth-rank elasticity tensor
$\mathbf{C}_{\mathrm{u}p}$: a symmetric second-rank tensor.

The latter can be interpreted as a stress-free configuration. The isomorphy condition (13) between the current and the elastic reference law takes the form

$$\mathbf{K}_p[\mathbf{C} - \mathbf{C}_{\mathrm{u}p}] = \mathbf{P}\mathbf{K}_0[\mathbf{P}^T\mathbf{C}\mathbf{P} - \mathbf{C}_{\mathrm{u}0}]\mathbf{P}^T \tag{42}$$

which serves to determine $\mathbf{K}_p$ and $\mathbf{C}_{\mathrm{u}p}$ by their reference values $\mathbf{K}_0$ and $\mathbf{C}_{\mathrm{u}0}$ via the plastic transformation $\mathbf{P}$. For crystals, the elastic behavior must be expected to be anisotropic. It is therefore convenient to use a crystallographic vector base $\{\mathbf{g}_{pj}\}$ and its dual $\{\mathbf{g}_p^j\}$ to represent

$$\mathbf{K}_p = \mathrm{K}_p^{ijkl}\mathbf{g}_{pi} \otimes \mathbf{g}_{pj} \otimes \mathbf{g}_{pk} \otimes \mathbf{g}_{pl} \tag{43}$$

and

$$\mathbf{C}_{\mathrm{u}p} = \mathrm{C}_{\mathrm{u}pij}\, \mathbf{g}_p^i \otimes \mathbf{g}_p^j. \tag{44}$$

By defining a crystallographic reference base by $\{\mathbf{g}_{0j} := \mathbf{P}^{-1}\mathbf{g}_{pj}\}$ and its dual $\{\mathbf{g}_0^j := \mathbf{P}^T\mathbf{g}_p^j\}$ and exploiting the isomorphy condition, one can easily show that $\mathbf{K}_p$

and $\mathbf{C}_{up}$ have the same components with respect to $\{\mathbf{g}_{pj}\}$ and $\{\mathbf{g}_p^j\}$ as $\mathbf{K}_0$ and $\mathbf{C}_{u0}$ have with respect to $\{\mathbf{g}_{0j}\}$ and $\{\mathbf{g}_o^j\}$. In other words, $\mathbf{P}$ transforms the anisotropy directions being characteristic for $\mathbf{K}_0$ into those of $\mathbf{K}_p$. Or, $\mathbf{P}$ can be interpreted as an identification of material line elements in a crystalline point that play the same role for the anisotropic elastic behavior in two different elastic ranges. Numerical applications of this model in (a slightly rate-dependent version) to single and polycrystalline materials are given in Bertram and Kraska (1995a,b), Bertram et al. (1997a,b), Kraska and Bertram (1996), Böhlke et al. (1997).

4. The multiplicative decomposition

Most theories in the field of finite plasticity are based on the concept of *intermediate placements*.[6] The assumption behind this concept is that at any instant a material point can be (at least locally) unloaded via an elastic process. As a result, we obtain a stress-free or released placement called *intermediate*. Clearly, the intermediate placement can be subjected to a rigid body motion without generating stresses according to the usual invariance requirements. We will next show how this notion of an intermediate placement can be imbedded into the present framework.

Up to now, the reference placement did not play an important role. It was chosen arbitrarily once and for all. From now on, however, we restrict this freedom and choose the reference placement in such a way that the elastic reference law of our material point gives zero stresses

$$h_0(\mathbf{I}) = \mathbf{0} \tag{45}$$

This assumption is not very restrictive, as we do not demand that the material can always or even once reach this state via elastic unloading. From the isomorphy condition (13) then

$$h_p(\mathbf{P}^{-T}\mathbf{P}^{-1}) = \mathbf{0} \tag{46}$$

holds for any elastic range, i.e. $\mathbf{P}^{-T}\,\mathbf{P}^{-1}$ describes the current stress-free configuration, again without saying that it is necessarily contained in the current elastic range. Indeed, there are materials where this is not always the case (because of a strong Bauschinger effect[7]).

If we identify the current local *intermediate placement* by

$$\mathbf{F}_p := \mathbf{P}^{-1} \in \mathscr{L}_{in} \tag{47}$$

[6]In using the word *placement* instead of *configuration*, we follow Noll (1972). A configuration is an equivalence class of placements that differ only by a rigid body motion. We believe that these expressions come closer to the common meaning of these words. If linearized in a material point, the deformation gradient is a *local placement*, whereas the right Cauchy–Green tensor describes the *local configuration*. Consequently, we will speak of reference and intermediate *placements* instead of *configurations*.

[7]Mandel (1974) considers this situation and calls it a *virtual unloading*.

and

$$\mathbf{F}_e := \mathbf{F}\mathbf{F}_p^{-1} \in \mathscr{Lin} \tag{48}$$

as the plastic and elastic parts of the deformation gradient $\mathbf{F}$, we obtain the *multiplicative decomposition*

$$\mathbf{F} = \mathbf{F}_e\mathbf{F}_p \tag{49}$$

(see Lee 1969, Mandel, 1971, 1973, 1974). We determine the Cauchy stress by (1, 4, 13, 47) as

$$\mathbf{T} = \mathbf{F}h_p(\mathbf{C})\mathbf{F}^T = \mathbf{F}_e h_0(\mathbf{C}_e)\mathbf{F}_e^T \tag{50}$$

with $\mathbf{C}_e := \mathbf{F}_e^T\mathbf{F}_e$. For $\mathbf{F} \equiv \mathbf{F}_p$, $\mathbf{C}$ coincides with $\mathbf{P}^{-T}\mathbf{P}^{-1}$ and, thus, $\mathbf{T} \equiv \mathbf{0}$. Hence $\mathbf{C}_e \equiv \mathbf{I}$ describes a stress-free configuration. If the inelastic reference law is derived from a potential w_0, then

$$\mathbf{T} = 2\rho\mathbf{F}_e \operatorname{grad} w_0(\mathbf{C}_e)\mathbf{F}_e^T. \tag{51}$$

By assuming that $\mathbf{P}$ is unimodular, we finally obtain

$$\mathbf{T} = 2\det(\mathbf{F}_e)^{-1}\rho_0\mathbf{F}_e \operatorname{grad} w_0(\mathbf{C}_e)\mathbf{F}_e^T \tag{52}$$

which corresponds to Eq. (18) of Lee (1969). In this paper, however, Lee postulated the symmetry of $\mathbf{F}_e$. This can be achieved by the polar decomposition of

$$\mathbf{F}\mathbf{P} =: \mathbf{V}_e\mathbf{R}_e \tag{53}$$

into $\mathbf{V}_e \in \mathscr{Sym}$ and $\mathbf{R}_e \in \mathscr{Orth}^+$. By defining instead of Eqs. (47) and (48)

$$\mathbf{F}_e := \mathbf{V}_e \tag{54}$$

and

$$\mathbf{F}_p := \mathbf{R}_e\mathbf{P}^{-1}, \tag{55}$$

Eq. (49) again holds. Then, instead of (52) we obtain

$$\mathbf{T} = \mathbf{F}_e\mathbf{R}_e h_0(\mathbf{R}_e^T\mathbf{C}_e\mathbf{R}_e)\mathbf{R}_e^T\mathbf{F}_e^T. \tag{56}$$

If h_0 is an isotropic tensor function, this expression coincides with Eq. (50). This is the case if the elastic reference law h_0 describes isotropic elasticity with respect to some undistorted isotropic configuration $\mathbf{C}_e \equiv \mathbf{I}$.

Under these identifications, however, $\mathbf{F}_p$ (or its orthogonal part) varies even under rigid body rotations. In our theory this has no influence, as our (material) variables

remain unaltered under rigid rotations. Moreover, the symmetry of $\mathbf{F}_e$ is not needed, as its orthogonal part does not enter $\mathbf{C}_e$ by definition.

Let us turn back again to the general anisotropic case and the identifications Eqs. (47) and (48). If we calculate the second Piola–Kirchhoff tensor with respect to the (local) intermediate placement we obtain

$$\mathbf{T}_i^{\mathrm{2PK}} := \det(\mathbf{F}_e)\mathbf{F}_e^{-1}\mathbf{T}\mathbf{F}_e^{-T} = \frac{\rho_0}{\rho} h_0(\mathbf{C}_e) \tag{57}$$

or

$$\mathbf{T}_i^{\mathrm{2PK}} = 2\rho_0 \operatorname{grad} w_0(\mathbf{C}_e) \tag{58}$$

in the case of hyperelastic ranges. This corresponds to Mandel's suggestion. Note that this equations does not involve any plastic variable anymore, but only a symmetric elastic configuration tensor. Moreover, the elastic reference law h_0 or the reference potential w_0 are independent of the plastic deformation. However, the underlying intermediate placement, which Mandel called *isoclinic*, continuously varies in time and space during plastic processes. In Mandel's format, neither $\mathbf{F}_p$ nor $\mathbf{F}_e$ are assumed to be symmetric (in contrast to Lee's). However, only the symmetric part of $\mathbf{F}_e$ enters Eq. (57). The isoclinic placement cannot be arbitrarily rotated. It can only be modified by arbitrary symmetry transformations from the symmetry group of h_0, same as our plastic transformation.[8]

It was Green and Naghdi's 1971[9] objection against the multiplicative decomposition that "for objectivity requirements the intermediate configuration should be rotatable under any time-dependent orthogonal transformations (Euclidean transformation)." Within the present context we clearly see that this postulate is too strong and not backed by objectivity requirements. Instead, it would lead to (elastic) isotropy, a property which has nothing to do with objectivity. However, the isoclinic placement can be rotated by any *constant* orthogonal tensor $\mathbf{Q}$. We would again obtain an isoclinic placement with elastic law

$$\underline{h}_0(\mathbf{C}) = \mathbf{Q}\, h_0(\mathbf{Q}^T\mathbf{C}\mathbf{Q})\mathbf{Q}^T. \tag{59}$$

If the elastic law is isotropic, this transformation does not affect the elastic law $\underline{h}_0 \equiv h_0$.

Mandel's format contains full generality with respect to material anisotropy, and is therefore capable to describe crystal plasticity (in contrast to Lee, who restricted himself to the isotropic case).

It is worth noting that with their choices of elastic laws Eqs. (52) and (58), both Lee's and Mandel's theories fulfill the isomorphy requirement of our Assumption 2. Indeed, there are few theories in the field of plasticity that do not use this property (see Krawietz, 1986; Krempl, 1994).

By using Eqs. (47)–(49), we obtain for the velocity gradient

[8]This argument can be found in Dashner (1986) after Eq. (15).

[9]See also Casey and Naghdi (1980).

$$\mathbf{L} = \mathbf{F}^{\bullet}\mathbf{F}^{-1} = \mathbf{L}_e + \mathbf{L}_p \tag{60}$$

with the definitions of Lee

$$\mathbf{L}_e := \mathbf{F}_e^{\bullet}\mathbf{F}_e^{-1} \tag{61}$$

and

$$\mathbf{L}_p := \mathbf{F}_e\mathbf{F}_p^{\bullet}\mathbf{F}_p^{-1}\mathbf{F}_e^{-1} = -\mathbf{F}\mathbf{P}^{\bullet}\mathbf{P}^{-1}\mathbf{F}^{-1} \tag{62}$$

the latter being the push forward of $-\mathbf{P}^{\bullet}\mathbf{P}^{-1}$. None of them is a gradient (not even under homogeneous deformations). As $\mathbf{P}$ is non-symmetric in general, none of the two parts is symmetric in general. Clearly, neither the skew part of $\mathbf{L}_e$ nor of $\mathbf{L}_p$ directly contribute to the stress-power (2). However, the non-symmetric part of $\mathbf{P}$ enters the elastic law for the stresses through (13) and thus does indirectly influence the stress power.

If we plug the evolution law for the plastic transformation into (62) we obtain

$$\mathbf{L}_p = -\mathbf{F}\, p(\mathbf{P}, \mathbf{C}, \mathbf{Z}_p, \mathbf{C}^{\bullet})\mathbf{F}^{-1} =: l_p(\mathbf{P}, \mathbf{F}, \mathbf{Z}_p, \mathbf{F}^{\bullet}) \tag{63}$$

so that $\mathbf{L}_p$ is fully determined by a material law. Similarly, the same holds for

$$\mathbf{L}_e =: l_e(\mathbf{P}, \mathbf{F}, \mathbf{Z}_p, \mathbf{F}^{\bullet}) \tag{64}$$

We see neither a theoretical nor a numerical advantage in decomposing $\mathbf{L}_p$ into its symmetric and skew parts (the latter is often called *plastic spin*). A single evolution equation gives both parts simultaneously, as we already have examplified for crystal plasticity.

5. The additive decomposition

Another constitutive decomposition of the deformations into elastic and plastic parts was suggested by Green and Naghdi (1965) with respect to Green's strain tensor

$$\mathbf{E}^G := {}^1\!/_2(\mathbf{C} - \mathbf{I}) = \mathbf{E}^e + \mathbf{E}^p \in \mathrm{Sym} \tag{65}$$

In the original work, $\mathbf{E}^p$ is introduced as a primitive concept, and $\mathbf{E}^e$ definded as the difference $\mathbf{E}-\mathbf{E}^p$. The stress law is assumed to be

$$\mathbf{T}^{2\mathrm{PK}} = g(\mathbf{E}^e, \mathbf{E}^p) \in \mathrm{Sym} \tag{66}$$

for the 2. Piola–Kirchhoff tensor with respect to an (arbitrary) reference placement. Later we find [Casey and Naghdi, 1980, Eq. (2)]

$$\mathbf{E}^p := {}^1\!/_2(\mathbf{F}_p^T\mathbf{F}_p - \mathbf{I}) \in \mathscr{Sym} \tag{67}$$

Consequently,

$$\mathbf{E}^e = \mathbf{F}_p^T\, {}^1\!/_2(\mathbf{F}_e^T\mathbf{F}_e - \mathbf{I})\mathbf{F}_p \in \mathscr{Sym} \tag{68}$$

By using the same identification Eq. (47) as before, we get

$$\mathbf{E}^p = {}^1\!/_2\,(\mathbf{P}^{-T}\mathbf{P}^{-1} - \mathbf{I}) \tag{69}$$

which has zero value for the current unloaded configuration. By this identification we obtain

$$\begin{aligned} \mathbf{T}^{2\mathrm{PK}} &= \frac{\rho_0}{\rho}\mathbf{F}_p^{-1}h_0[\mathbf{F}_p^{-T}(2\mathbf{E}^e + 2\mathbf{E}^p + \mathbf{I})\mathbf{F}_p^{-1}]\mathbf{F}_p^{-T} \\ &=: k(\mathbf{F}_p, \mathbf{E}^e) \end{aligned} \tag{70}$$

which is not a special form of Eq. (66). Again we see that a symmetric variable such as $\mathbf{E}^p$ is not sufficient to describe full anisotropy. Only in the isotropic case, $\mathbf{F}_p$ can be taken as symmetric and Eq. (70) can be brought into the form Eq. (66) by substituting

$$\mathbf{F}_p \equiv (2\mathbf{E}^p + \mathbf{I})^{1/2} \in \mathscr{Sym} \tag{71}$$

In this special case, however, Eq. (66) is more general than the specific form Eq. (70), as our Assumption 2 does not automatically hold. If it does hold, as is commonly assumed, Eq. (66) must have the form Eq. (70).

6. Conclusions

Our format for the elasto-plastic model is based on the following essentials:

- the material is rate-independent;
- deformation processes are the input variables, stress-processes are output-variables;
- the material has elastic ranges with isomorphic elastic laws.

The latter assumption is considered as fundamental for most of the theories in the field, although seldom mentioned. Based on these assumptions, an elastic-plastic model has been established that consists of (i) the elastic reference law h_0, (ii) the flow rule Π, (iii) the hardening rule Z, and (iv) the yield criterion φ. The entire theory is formulated in material or Lagrangean variables and, thus, identically fulfills the principle of material frame-invariance. With this general framework, we are able to imbed the main theories in the field and to investigate their validity and generality.

The use of reference placements for such materials and, as a consequence, that of deformation gradients with respect to them, gave rise to a lot of misunderstanding and confusion in finite plasticity. However, Noll (1972) has shown that material theory can be concisely formulated without reference placements (intrinsic description). For the present theory only one reference placement has been used which can be arbitrarily chosen as it does not play any particular role. Instead of deformation gradients we used the configuration tensor $\mathbf{C}$ which is immune to all rigid rotations of the body and changes of observer.

We also did not make use of the notion of elastic unloading into intermediate placements, which caused more confusion than insight in the past because of undefined rotations. Neither do we use constitutive decompositions into elastic and plastic part of strains or strain-rates. The split of the rate of our plastic variable $\mathbf{P}^{\bullet}\mathbf{P}^{-1}$ (or likewise of $\mathbf{L}_p$), into symmetric and skew parts does not bring any benefit, as one single evolution function for the entire variable is needed.

Because of certain symmetry assumptions, Lee's theory is limited to isotropic behavior. The same result holds for Green and Naghdi's theory, although quite different in nature. Only Mandel's format covers full anisotropy. As a general result we can state that

- in the anisotropic case a non-symmetric internal variable ($\mathbf{P}$, $\mathbf{F}_p$, etc.) is needed, and, consequently, an evolution law which also determines the skew parts of its rate, i.e. the plastic spin;
- in the isotropic case, however, the orthogonal parts of the plastic variable and the skew parts of their rate forms can be ruled out.

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