Bertram , A. ; Sampaio, R. : A Constitutive Theory for Friction

Frictional laws, describing the contact between continuous bodies, are *constitutive equations*. Within a constitutive theory, we can summit frictional laws to the usual principles (*determinism, objektivity, local action, dissipation*), apply *material isomorphisms* and *symmetry transformations*, and categorize them within the usual classes of materials. Our purpose is to identify the general form of a frictional law as a *material system* in the sence of [2] and thus, obtain all the mentioned concepts. In doing this, we are confronted with the following problems:

- There are always *two* distinct bodies involved in friction. The surface points being coupled may vary all the time.
- In order to describe *sticking* and *sliding* and *no-contact*, the velocity field can be discontinuous with respect to time.
- The cases of *rigid* or *elastic* behaviour are irrelevant in friction.
- Coulomb´s law must fit into the theory.

I. Geometry of contact

We consider a solid body eventually being in contact with a surface, which may belong to another body. So, let the *body* B be a 3-dimensional differentiable open manifold. The *boundary* of B is dB . On the other hand, let the *surface* S be a 2-dimensional differentiable closed manifold without boundary. The 2-dimensional tangential space (without inner product) to dB is $T_X dB$, and $T^*_X dB$ its dual. E is the 3-dimensional *Euclidian space* being endowed with its *translational space* V , a 3-dimensional vector space with inner-product " • ".The *time interval* T is assumed to be an open real interval with elements called *instants*.

A *motion* of the body is a regular time-dependent imbedding $\mu : (B \cup dB) \times T \rightarrow E$ being differentiable on B, one-sided differentiable on dB, and piecewise differentiable on T. The region occupied by the body B at some instant t is the set $B_t := \mu(B,t) \subset E$ and, analogously, $dB_t := \mu(dB,t) \subset E$. The differential of μ in $X \in dB$ at $t \in T$ is

$\mathbf{F}: T_X dB \rightarrow T_x dB_t \subset V$, being invertible. The past and future *velocities* are the vector fields $\mathbf{w}^-$ and $\mathbf{w}^+$, respectively, being defined on dB_t.

A *motion* of S is a regular time-dependent imbedding $\lambda: S \times T \rightarrow E$ being spacially differentiable and piecewise differentiable with respect to time. The spacial region occupied instantaneously by S is $S_t := \lambda(S,t) \subset E$. In order to exclude *penetration*, we will always assume, that $S_t \cap B_t = \emptyset$. If, eventually, the boundary dB_t and the surface S_t coincide, then they are in contact. The *contact area* is $C_t := S_t \cap dB_t$. Of course, this set might be empty at instants of no contact. The pull-back of C_t on dB is the closed set $C_o(t) := \{X \in dB \mid \mu(X,t) \in C_t\}$. The past and future velocities of S are $\mathbf{u}^-$ and $\mathbf{u}^+$, respectively. The *relative velocities* are defined as $\mathbf{v}^- := \mathbf{w}^- - \mathbf{u}^-$ and $\mathbf{v}^+ := \mathbf{w}^+ - \mathbf{u}^+$. The following cases are possible for all instants and surface points:

- past or future *sticking*:	$\mathbf{v}^{-,+} = \mathbf{o}$
- past or future *sliding*:	$\mathbf{v}^{-,+} \neq \mathbf{o}$ and $\mathbf{v}^{-,+} \cdot \mathbf{n} = 0$
- starting *contact*:	$\mathbf{v}^-$ not defined and $\mathbf{v}^+$ defined
- loosing contact:	$\mathbf{v}^-$ defined and $\mathbf{v}^+$ not defined

The following cases are *not* possible:

- past or future *penetration*:	$\mathbf{v}^{-,+} \cdot \mathbf{n} \neq 0$

II. Constitutive theory

All the following concepts are local in the sence, that the footpoint $X \in dB$ or $x \in dB_t$ is hold fixed. Let us assume that the body consists of a simple solid material without internal constraints in the sense of [2] and [4]. It may be elastic or inelastic, isotropic or anisotropic, homogenous or non-homogeneous. We consider processes of surface stresses of duration $d \in \mathbb{R}$, starting at a fixed initial state at t_i of the form $\mathbf{s} : T \supset [t_i, t_i + d] \rightarrow V$. At each instant, one can decompose $\mathbf{s}$ into its normal and tangential part $\mathbf{s} = \sigma\mathbf{n} + \boldsymbol{\tau}$. Then we pull $\boldsymbol{\tau}$ back to the cotangent space $\boldsymbol{\tau}_o = \mathbf{F}^* \boldsymbol{\tau} \in T^*_X dB$. So each s-process may be equivalently be considered as a $(\sigma, \boldsymbol{\tau}_o)$-process with values in $\mathbb{R} \times T^*_X dB$. The set of all possible processes of this type forms the *process class* Ω_o, which is assumed to be closed under restriction and continuation (see [2],

p. 103). Let $\mathbf{t} \neq \mathbf{o} \in T_x dB_t$. Then we call the vector-set $\{\mathbf{v} \in T_x dB_t | \ \mathbf{v} = a\ \mathbf{t}\ ,\ a > 0 \in \mathbb{R}\}$ the *t-ray*. The set of all t-rays, as $\mathbf{t}$ runs over the unit-ball of $T_x dB_t$, is denoted by R. The pull back of the relative velocity is defined as the tangent vector $\mathbf{v_o} = \mathbf{F}^{-1}(\mathbf{v}) \ \in T_X dB$, and

$$D_o := \{\varnothing\} \cup \mathbf{F}^{-1}(T_x dB_t) \cup \mathbf{F}^{-1}(R) = \{\varnothing\} \cup T_X dB \cup R_o \ .$$

A *constitutive equation for friction* is a mapping $f : \Omega_o \to D_o$, such that the push-forward of the values $f(\sigma, \tau_o)$ can be interpreted as the set of *all possible future relative velocities* of the point at the end of a certain (σ, τ_o)-process. So, the values of f can be

1) the empty set $\varnothing$, if there is eventually *no contact;*
2) the zero vector $\mathbf{o} \in T_X dB$, if there occurs *sticking;*
3) a t-ray, in the case of undetermined *sliding* in the direction of $\mathbf{F}\,\mathbf{t}$;
4) a tangent vector $\mathbf{v_o} \neq \mathbf{o} \in T_X dB$ in the case of *sliding* with relative velocity $\mathbf{F}\,\mathbf{v_o}$.

The mapping f, being a constitutive function, is submitted to the principles of rational mechanics. The principle of *determinism* and the principle of *local action* are a-priori fullfilled by the definitions. In order to apply the principle of *material objectivity,* we state that the relative velocity, the normal stress and the tangential stress turn out to be objective, whereas its pull-backs and the (σ, τ_o)-processes are invariant. Thus, the *constitutive equation for friction* f automaticly obeys the principle of material objectivity, if, and only if it is the same for all observers (*forminvariance*).

The *principle of dissipation* can be satisfied, if for all admissible s-processes

$$\mathbf{s} \cdot \mathbf{v} = \ \tau \cdot \mathbf{v} \ = \ \mathbf{F}\,\tau_o \ \cdot \ \mathbf{F}^{-*}\,\mathbf{v_o} \ = \ < \tau_o \ , \mathbf{F}^{*}\,\mathbf{F}^{-*}\,\mathbf{v_o} > \ = \ < \tau_o \ , \ \mathbf{v_o} > \quad \leq 0$$

holds for all $\mathbf{v_o} \in f(\sigma, \tau_o)$. Clearly, this is identically satisfied by the values $\varnothing$ and $\mathbf{o}$, as in the case of no-contact and sticking, there is no dissipation due to friction.

The triple (Ω_o, D_o, f) is a *material system* in the sence of [2]. By this identification, we can apply all the concepts of material theory to friction, such as *states, aging, revertibility, rate-independence, material isomorphy* and *symmetry,* and *isotropy.*

III. Example: a material system of the Coulomb-type

The most well-known and widely applied frictional law is due to COULOMB [3], based on a suggestion by AMONTONS [1]. Let $\sigma_a, \mu_s \geq \mu_k \in \mathbb{R}$ be three positive constants. The stress process class contains all processes starting at some $\mathbf{s}$ and being arbitrarily composed out of the three following types of processes:

(1) no contact: $\mathbf{s} = \mathbf{o} = (\sigma = 0, \boldsymbol{\tau}_o = \mathbf{o})$

(2) sticking: such that $|\boldsymbol{\tau}_o| \leq \sigma_a + \mu_s \sigma$

(3) sliding: $|\boldsymbol{\tau}_o| = \mu_k \sigma$

such that segments of type (3) are following those of type (2), only if equality was reached at the end of the stick-segment. The frictional constitutive equation has the following values

(1) the empty set $\emptyset$ during segments of no contact;

(2) the zero vector $\mathbf{o}$ during segments of sticking; and

(3) the $\boldsymbol{\tau}_o$-ray during sliding.

References

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