Materials with Isomorphic Elastic Ranges - an Approach to Finite Plasticity

Albrecht Bertram

Otto-von-Guericke-Universität Magdeburg, Institut für Mechanik, Universitätsplatz 2, D-39106 Magdeburg, Germany
bertram@mb.uni-magdeburg.de

Abstract
After three decades of intense research activities in the field, finite plasticity is still far from being a unified theory based on a firm theoretical ground. Therefore, we intent to introduce a general theory of plasticity based on two constitutive assumptions, (*i*) the existence of elastic ranges, and (*ii*) their elastic isomorphy. Classical theories are shown to fit into this general format after appropriate identifications.

Introduction
During the past three decades, there have been various suggestions to generalize the elements of classical plasticity theory to large deformations. These suggestions differ even in their most fundamental concepts (Naghdi 1990), and it is by no means an easy task to compare them and to show their ranges of agreement and disagreement, their limitations, and their validity. This is due to at least two facts. Firstly, in plasticity it is rather common to use intuitive notions instead of mathematically and physically precisely defined concepts. And secondly, there is no general theory or framework, in which we could imbed all these suggestions and compare them on such a firm ground.

General Plastic Behavior
The present paper is intented to establish such a framework for plasticity theories. For that purpose, we try to work out the most common features of what one understands by „plastic behavior". As such we chose:

- the existence of elastic ranges;
- the occurence of permanent deformations during yielding;
- the rate-independence of the entire model;
- the independence of the elasticities from prior plastic deformations (no induced elastic anisotropy).

We do not intent to say that all plastic materials have these properties. But we claim that a general theory on plasticity should be capable to describe such behavior. We base our approach on a strict distinction between independent and dependent variables. In the mechanical theory, deformation processes are independent, stresses dependent. In the thermodynamical theory, the processes of the local temperature, of its gradient, and of the deformation are independent, whereas the free energy, the entropy, the heat flux, and the stresses are considered as dependent variables. In the current context, however, we limit our considerations to the purely mechanical theory for the sake of brevity. For the thermodynamical theory the reader is referred to Bertram (1999a).

The material class under consideration is assumed to obey the principle of determinism in the following (reduced) form which identically fulfills the requirement of material objectivity: *the* *deformation process* C(t) (right Cauchy-Green-tensor) *determines the final material stress tensor* S(t).

Materials with Isomorphic Elastic Ranges

While the preceeding assumption is valid for almost all materials, the following constitutive assumptions are specific for plastic materials. We define an elastic range within the configuration space being formed by candidates for C, i. e., all symmetric and positive definite tensors. An *elastic range* is established by a set $\{\mathcal{E}_p, h_p\}$ with

- $\mathcal{E}_p$ being a path-connected closed subset of the configuration space, and
- h_p being an elastic law

$$\mathbf{S} = h_p(\mathbf{C}) \quad \forall\, \mathbf{C} \in \mathcal{E}_p .$$

At any instant, the material is assumed to be located within some elastic range. If the further configuration process remains in $\mathcal{E}_p$, the material behaves elastically, i. e., the stresses are determined by the current configuration C through h_p . If the configuration process penetrates the boundary of the current elastic range $\mathcal{E}_p$, the material necessary changes the elastic range continuously.

The second constitutive assumption concerns the elastic behavior of the material within different elastic ranges. For most crystalline materials, the stresses are determined by the deformation of the lattice. If we consider a material point within two states, where one is obtained from the other by large deformations, then we nevertheless expect the stresses to be equal, if the lattice (not the material) is equally deformed. We give this notion of equal elastic behavior a mathematically precise form by assuming the existence of an elastic isomorphism $\mathbf{P}_{12}$, an invertible unimodular tensor or rank 2, such that the elastic laws of two elastic ranges, h_1 and h_2, can be related according to

$$h_2(\mathbf{C}) = \mathbf{P}_{12}\; h_1(\mathbf{P}_{12}^T\, \mathbf{C}\, \mathbf{P}_{12})\, \mathbf{P}_{12}^T$$

(see Bertram 1989). The index T denotes the transposition of a tensor. Note that this isomorphy concept is a purely elastic one. Nevertheless, it is the appropriate tool to describe the effect of plastic deformations on the elastic behavior of different elastic ranges. If all elastic ranges that a material point could potentially pass through, are mutually isomorphic, then we can freely choose one of them, say $\{\mathcal{E}_0, h_0\}$, and transform any other one $\{\mathcal{E}_p, h_p\}$ by means of the isomorphy condition to the referential one

$$h_p(\mathbf{C}) = \mathbf{P}\; h_0(\mathbf{P}^T\, \mathbf{C}\, \mathbf{P})\, \mathbf{P}^T.$$

We call P *plastic transformation* (Wang *et al.* 1974), which varies (only) during states of yielding. Thus, the time-dependence of the function h_p during yielding is reduced to the time dependence of one tensorial variable P. It can be easily shown (Bertram 1992) that P is unique only up to symmetry transformations, i.e., if A is element of the symmetry group of h_0 , then PA also serves as a plastic transformation from h_0 to h_p . An important consequence of this fact is the following. If, and only if h_0 is isotropic, then the orthogonal part of P is arbitrary and can be ruled out by normalization. Thus, only in the isotropic case, the plastic transformation can be assumed to be symmetric. This has some consequences for the issue of *plastic spin*.

With these two assumptions alone we are able to relate some classical approaches to finite plasticity to the present one.

The Multiplicative Decomposition of Lee and Mandel

If we identify

$$\mathbf{F}_p := \mathbf{P}^{-1}$$

and

$$\mathbf{F}_e := \mathbf{F}\,\mathbf{F}_p^{-1}$$

we obtain the multiplicative decomposition of the deformation gradient

$$\mathbf{F} = \mathbf{F}_e \mathbf{F}_p .$$

It should be emphasized, however, that other identifications, different from the above, also lead to analoge results. If the reference configuration is chosen as stress free with respect to h_0, i. e.,

$$h_0(\mathbf{I}) = \mathbf{0}\,,$$

then we obtain in connection with the isomorphy condition

$$\mathbf{T} = \mathbf{F}_e\, h_0(\mathbf{F}_e^T \mathbf{F}_e)\, \mathbf{F}_e^T$$

which corresponds to Lee's eqn. 18 (Lee 1969).

If we calculate the 2. Piola-Kirchhoff stresses with respect to the isoclinic (local) placement defined by $\mathbf{F}_p$, we obtain

$$\mathbf{T}^{2PK}{}_i = \frac{\rho_0}{\rho}\, h_0(\mathbf{C}_e)$$

which corresponds to Mandel's suggestion with $\mathbf{C}_e := \mathbf{F}_e^T \mathbf{F}_e$ (Mandel 1971).

The Additive Decomposition of Green and Naghdi

If we identify

$$\mathbf{E}^p := \tfrac{1}{2}\,(\mathbf{P}^{-T}\mathbf{P}^{-1} - \mathbf{I})$$

and

$$\mathbf{E}^e := \mathbf{E} - \mathbf{E}^p,$$

the additive decomposition of Green's deformation tensor

$$\mathbf{E} := \tfrac{1}{2}\,(\mathbf{C} - \mathbf{I}) = \mathbf{E}^e + \mathbf{E}^p$$

is accomplished. Again by the isomorphy we obtain for the 2. Piola-Kirchhoff stresses

$$\mathbf{T}^{2PK} = \frac{\rho}{\rho_0} \mathbf{P}\, h_0[\mathbf{P}^T (2\mathbf{E}^e + 2\mathbf{E}^p + \mathbf{I})\, \mathbf{P}]\, \mathbf{P}^T,$$

which corresponds to Green and Naghdi's suggestion (Green *et al.* 1965)

$$\mathbf{T}^{2PK} = g(\mathbf{E}^e, \mathbf{E}^p)$$

only if the elastic law is isotropic and, hence, **P** can be taken as symmetric. In this case,

$$\mathbf{P} = (2\,\mathbf{E}^p + \mathbf{I})^{-1/2}.$$

Conclusion

The theory of materials with isomorphic elastic ranges has been very briefly depicted in the present paper. It can be found in details in Bertram (1992, 1998, 1999a, b). It has been applied to single and polycrystals in Bertram *et al.* (1995, 1997).

Some of its characteristics are

- no constitutive assumption on the decomposition of the deformation into elastic and plastic parts has been made,
- the choice of the reference placements plays no role and remains free,
- the intire model is formulated in material variables and, thus, is consistent with objectivity requirements,
- all constitutive ingredients can be either isotropic or anisotropic.

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