

Albrecht Bertram · Arnold Krawietz

On the introduction of thermoplasticity

Received: 7 October 2011 / Revised: 17 February 2012 / Published online: 17 July 2012
© Springer-Verlag 2012

Abstract The introduction of a format for thermoplasticity is given based on the quantities that are measurable, at least principally. The important assumption is that of equal thermoelastic behaviour within all elastic ranges. For simplicity, the theory is restricted to small deformations. A simple example is given to demonstrate the results. The question of uniqueness of the thermodynamical variables is discussed.

1 Introduction

If yielding occurs in a plastic material, then the work done by the stresses on the plastic strain is partly transformed into heat, while some other part may go into reversible or irreversible changes of the microstructure. This can lead to larger changes of temperature, at least locally, which influence the elastic behaviour as well. For this reason, a thermomechanical theory is needed, which will also offer the possibility to investigate the restrictions caused by the second law to this particular class of materials so as to obtain a thermodynamically consistent theory of plasticity.

In the literature, there are many suggestions for the introduction of thermodynamics into plasticity ([1, 2, 5, 12, 14, 16, 18, 20, 22, 23], to mention just a few of them). Most of these authors make assumptions upon the form of the free energy. By the second law of thermodynamics in the form of the Clausius–Duhem-inequality, one can then find restrictions on the constitutive equations. In some cases, incorrect conclusions are drawn like, for example, the positivity of the work of stress on the plastic deformations. In all cases, the question of the uniqueness of the elastic and plastic parts of the free and internal energy and the entropy arises and remains often without answer.

In the present paper, it is intended to introduce only assumptions to the constitutive theory by physical arguments, and not just motivated by their consequences. The physicality of these assumptions is closely related to the question of measurability of certain quantities. To give an example, the free energy is certainly not a quantity that can be directly measured in experiments, neither is the entropy, in contrast to (the rate of) the internal energy.

The principal physical assumption which we will make is that the thermoelastic behaviour within the different elastic ranges is unaffected by plastic deformations. The elasticities, the coefficient of heat conduction, etc. are considered as material constants, and not as being path-dependent variables of the foregoing deformation process. This is a common assumption in plasticity theory, although not often stated in this clearness.

A. Bertram (✉)
IFME, University of Magdeburg, Universitätsplatz 2, 39106 Magdeburg, Germany
E-mail: albrecht.bertram@ovgu.de

A. Krawietz
Hildburghäuser Str. 241b, 12209 Berlin, Germany
E-mail: krawietz@t-online.de

By this assumption, we are able to obtain expressions for the dependent thermodynamic variables, which form necessary and sufficient conditions for this assumption to hold.

Our approach substantiates the additive decomposition of the internal energy and the entropy into an elastic and a plastic part. While the elastic and plastic part of the internal energy as well as the elastic part of the entropy are determined up to insignificant additive constants by the observable material behaviour, the plastic part of the entropy is only vaguely restricted by the dissipation inequality.

For the entire paper, we use the format of (geometrically) linear plasticity, being well aware of the limited range of validity or applicability of such theories. On the other hand, the geometrically linear mechanical theory of plasticity is much better understood than the nonlinear theory. However, in this paper, we do not want to address to the open questions of the theory of large plastic strains,¹ but limit our concern to the thermodynamical problems, which alone are difficult enough.

2 The theory of elastoplasticity

Before we go into the thermoplastic theory, we want to recall the classical, that is, geometrically linear *mechanical* theory of plasticity that consists of the following ingredients:

1. an *additive decomposition* of the linear strain tensor \mathbf{E} into an elastic and a plastic part

$$\mathbf{E} = \mathbf{E}_e + \mathbf{E}_p \quad (1)$$

2. an *elastic law* taken as the linear Hooke's law, connecting the elastic strain tensor and the stress tensor

$$\mathbf{T} = \mathbf{C} \cdot \mathbf{E}_e \quad (2)$$

by use of a fourth-order elasticity tensor \mathbf{C} with the usual symmetries. Each multiplication point stands for one contraction in this text. This law can be isotropic or anisotropic. The important underlying assumption (which practically all authors use in this context, even if not mentioned) is that the elasticities are constant, that is, not affected by plastic deformations. This means that we totally neglect the influence of some evolving microstructure like, for example, a crystallographic texture on the elasticities.² This is justified if either the elastic anisotropy of the constituting crystals of a polycrystalline material is weak, or if the crystallographic texture does not vary too much. The latter should be generally valid in the current context since we restrict our analysis to small deformations.

3. a *yield limit* (yield criterion), which forms the limit of the current elastic range. While in the one-dimensional theory one needs only a scalar quantity, the yield stress σ_Y , in a three-dimensional theory we have to take into account the multiaxiality of the stress state. We therefore make the general ansatz for the yield criterion as

$$\varphi(\mathbf{T}, \mathbf{H}), \quad (3)$$

where \mathbf{H} are scalar or tensorial plastic variables describing the hardening behaviour of the material or, in other words, the influence of the preceding deformation on the yield limit, which is the kernel of this function:

$$\varphi(\mathbf{T}, \mathbf{H}) = 0 \quad (4)$$

(*yield condition*), while we assume

$$\varphi(\mathbf{T}, \mathbf{H}) < 0 \quad (5)$$

(only) in the interior of the elastic range. Necessary and sufficient conditions for the material to yield are the yield condition and the *loading condition*

$$\frac{\partial \varphi}{\partial \mathbf{E}} \cdot \dot{\mathbf{E}} > 0. \quad (6)$$

Here the upper dot stands for the material time derivative.

¹ For thermoplasticity of large deformations see [4,9,13,17,19,21,26,28], among others.

² Such an effect is described by Böhlke and Bertram [6].

4. a *flow rule* that determines the evolution of the plastic strain. A general ansatz for it would be a first-order ODE depending on practically all variables:

$$\dot{\mathbf{E}}_p = F(\mathbf{E}, \mathbf{E}_p, \mathbf{E}_e, \mathbf{T}, \mathbf{H}, \dot{\mathbf{E}}). \quad (7)$$

However, we can express the elastic and plastic strain \mathbf{E}_e and \mathbf{E}_p by the total strain \mathbf{E} and the stress \mathbf{T} by means of the foregoing Eqs. (1) and (2), so that a function of a reduced set of variables

$$\dot{\mathbf{E}}_p = f(\mathbf{E}, \mathbf{T}, \mathbf{H}, \dot{\mathbf{E}}) \quad (8)$$

is sufficient.

5. a *hardening rule* for the variable(s) \mathbf{H} , which is assumed to be of the same form as the flow rule

$$\dot{\mathbf{H}} = h(\mathbf{E}, \mathbf{T}, \mathbf{H}, \dot{\mathbf{E}}). \quad (9)$$

Classical examples of such hardening variables are a variable scalar yield stress σ_Y in the case of isotropic hardening, a tensorial back stress in the case of kinematical hardening, or other tensorial variables or even a combination of them describing distortional hardening.

These five constitutive laws establish a complete mechanical rate-independent format for elasto-plasticity. In the sequel, we will generalize this format by introducing the thermodynamical variables and laws.

3 Thermoplasticity

For this purpose, we have to enlarge the set of purely mechanical variables by the thermodynamical ones. These are the following local quantities:

ε	the specific <i>internal energy</i> (a time-dependent scalar)
\mathbf{q}	the <i>heat flux</i> per unit time and area (a time-dependent vector)
θ	the <i>absolute temperature</i> (a time-dependent positive scalar)
\mathbf{g}	the <i>temperature gradient</i> (a time-dependent vector)
η	the specific <i>entropy</i> (a time-dependent scalar).

By *specific* we always mean with respect to unit mass.

We use the *energy balance* (first law) in the local form

$$\rho \dot{\varepsilon} = \rho Q + \mathbf{T} \cdot \cdot \dot{\mathbf{E}} \quad (10)$$

with ρ being the mass density and Q the heat supply per unit mass and unit time, which results from irradiation r and conduction \mathbf{q} in the form

$$\rho Q = \rho r - \operatorname{div} \mathbf{q}. \quad (11)$$

By the introduction of the *Helmholtz free energy*

$$\psi := \varepsilon - \theta \eta \quad (12)$$

we obtain the second law in form of the *Clausius–Duhem-inequality*

$$\dot{\psi} + \eta \dot{\theta} - \frac{1}{\rho} \mathbf{T} \cdot \cdot \dot{\mathbf{E}} + \frac{\mathbf{q} \cdot \mathbf{g}}{\rho \theta} \leq 0. \quad (13)$$

Thus, the specific *dissipation*, which consists of the mechanical dissipation

$$\delta_m := \frac{1}{\rho} \mathbf{T} \cdot \cdot \dot{\mathbf{E}} - \dot{\psi} - \eta \dot{\theta} = \theta \dot{\eta} - Q \quad (14)$$

and the thermal dissipation

$$\delta_{th} := -\frac{\mathbf{q} \cdot \mathbf{g}}{\rho \theta}, \quad (15)$$

fulfils the *dissipation inequality*

$$\delta = \delta_m + \delta_{th} \geq 0. \quad (16)$$

In order to enlarge the mechanical plasticity theory to a thermomechanical one, we add the temperature and the temperature gradient to the list of independent variables, and the heat flux, the entropy and the internal energy or the free energy to the dependent variables.

Thermoplastic materials can be understood as materials with *internal variables*. The set of the internal variables contains in the case of plasticity the plastic strain and the hardening variables. We collect these internal variables, which may be tensors of arbitrary order, in a finite-dimensional vector \mathbf{Z} and note it as a second-order tensor. For this variable, we assume a rate-independent evolution equation in the general form

$$\dot{\mathbf{Z}} = z(\mathbf{E}, \theta, \mathbf{Z}, \dot{\mathbf{E}}, \dot{\theta}). \quad (17)$$

The set of additional constitutive equations for a thermomechanical material with internal variables is

$$\begin{aligned} \mathbf{T} &= T(\mathbf{E}, \theta, \mathbf{g}, \mathbf{Z}) \\ \mathbf{q} &= q(\mathbf{E}, \theta, \mathbf{g}, \mathbf{Z}) \\ \eta &= \eta(\mathbf{E}, \theta, \mathbf{g}, \mathbf{Z}) \\ \varepsilon &= \varepsilon(\mathbf{E}, \theta, \mathbf{g}, \mathbf{Z}) \quad \text{or} \quad \psi = \psi(\mathbf{E}, \theta, \mathbf{g}, \mathbf{Z}). \end{aligned} \quad (18)$$

Instead of elastic ranges, we will now deal with *thermoelastic ranges*. These are specified by a *yield criterion*

$$\varphi(\mathbf{E}, \theta, \mathbf{Z}) \quad (19)$$

which induces the *yield condition*

$$\varphi(\mathbf{E}, \theta, \mathbf{Z}) = 0 \quad (20)$$

and the *loading condition*

$$\frac{\partial \varphi}{\partial \mathbf{E}} \cdot \dot{\mathbf{E}} + \frac{\partial \varphi}{\partial \theta} \dot{\theta} > 0. \quad (21)$$

If not both conditions are simultaneously fulfilled, the event is (thermo)elastic and, hence, $\dot{\mathbf{Z}} \equiv \mathbf{0}$. Otherwise it is a plastic event or an event of yielding, in which \mathbf{Z} evolves according to Eq. (17).

We will next investigate the restrictions by the Clausius–Duhem-inequality for the constitutive equations. With the rate of the free energy Eq. (18), we obtain for the inequality (13)

$$0 \geq \frac{\partial \psi}{\partial \mathbf{E}} \cdot \dot{\mathbf{E}} + \frac{\partial \psi}{\partial \theta} \dot{\theta} + \frac{\partial \psi}{\partial \mathbf{g}} \cdot \dot{\mathbf{g}} + \frac{\partial \psi}{\partial \mathbf{Z}} \cdot \dot{\mathbf{Z}} + \eta \dot{\theta} - \frac{1}{\rho} \mathbf{T} \cdot \dot{\mathbf{E}} + \frac{\mathbf{q} \cdot \mathbf{g}}{\rho \theta}. \quad (22)$$

If we first consider only elastic events ($\dot{\mathbf{Z}} \equiv \mathbf{0}$), then there remains

$$0 \geq \left(\frac{\partial \psi}{\partial \mathbf{E}} - \frac{\mathbf{T}}{\rho} \right) \cdot \dot{\mathbf{E}} + \left(\frac{\partial \psi}{\partial \theta} + \eta \right) \dot{\theta} + \frac{\partial \psi}{\partial \mathbf{g}} \cdot \dot{\mathbf{g}} + \frac{\mathbf{q} \cdot \mathbf{g}}{\rho \theta}. \quad (23)$$

The exploitation of this inequality for states below the yield limit, where $\dot{\mathbf{E}}$ and $\dot{\theta}$ are not restricted, leads by standard arguments to the usual conditions from thermoelasticity, namely the independence of the free energy of the temperature gradient $\psi = \psi(\mathbf{E}, \theta, \mathbf{Z})$ instead of Eq. (18) and the *thermoelastic potential relations*

$$\mathbf{T} = \rho \frac{\partial \psi}{\partial \mathbf{E}} = T(\mathbf{E}, \theta, \mathbf{Z}), \quad (24)$$

$$\eta = - \frac{\partial \psi}{\partial \theta} = \eta(\mathbf{E}, \theta, \mathbf{Z}) \quad (25)$$

as well as the *heat conduction inequality*

$$\delta_{th} := -q(\mathbf{E}, \theta, \mathbf{g}, \mathbf{Z}) \cdot \frac{\mathbf{g}}{\rho \theta} \geq 0 \quad (26)$$

as necessary and sufficient conditions for the second law to hold in elastic events.

Because of continuity, these restrictions must also hold when reaching the yield limit. If yielding occurs, however, then the remainders of the Clausius–Duhem-inequality (22) are

$$0 \geq \frac{\partial \psi}{\partial \mathbf{Z}} \cdot \dot{\mathbf{Z}} + \frac{\mathbf{q} \cdot \mathbf{g}}{\rho \theta}. \quad (27)$$

This inequality must be valid for any value of the temperature gradient \mathbf{g} and, hence, for an arbitrarily large absolute value of $\mathbf{q} \cdot \mathbf{g}$. Since the first term of the inequality does not depend on \mathbf{g} , we obtain again the heat conduction inequality (26) and, in addition, since $\mathbf{q} \cdot \mathbf{g}$ may be zero, the residual dissipation inequality

$$\delta_m = -\frac{\partial \psi}{\partial \mathbf{Z}} \cdot \dot{\mathbf{Z}} \geq 0. \quad (28)$$

One can further specify the evolution equation (17) by assuming the existence of a *dissipation potential* $\Pi_p(\mathbf{T}, \theta, \mathbf{Y})$ (see [12,22]) with

$$\mathbf{Y} := -\frac{\partial \psi}{\partial \mathbf{Z}}, \quad (29)$$

such that

$$\dot{\mathbf{Z}} = \lambda \frac{\partial \Pi_p}{\partial \mathbf{Y}} \quad (30)$$

with a non-negative *plastic parameter* λ . Then Eq. (28) becomes

$$\mathbf{Y} \cdot \dot{\mathbf{Y}} \geq 0. \quad (31)$$

If $\mathbf{Y} \equiv \mathbf{0}$ lies in the interior of the current elastic range, a sufficient condition for it to hold is the convexity of the dissipation potential. However, subsequently we will not make use of such a potential.

We now decompose the vector of the internal variables \mathbf{Z} into the plastic strain \mathbf{E}_p and the remaining (hardening) variables which may be tensors of arbitrary order, here noted as a second order tensor \mathbf{H} , and call them *plastic variables*. For the *yield condition*, we now write

$$\varphi(\mathbf{E}, \theta, \mathbf{E}_p, \mathbf{H}) = 0, \quad (32)$$

while the loading condition remains valid in the form of Eq. (21). The yield criterion in the strain space can be easily transformed into the stress space by Eqs. (1) and (2) if this is preferred.

For the free energy, we write with Eq. (1)

$$\psi = \psi(\mathbf{E}, \theta, \mathbf{E}_p, \mathbf{H}) = \psi(\mathbf{E}_e + \mathbf{E}_p, \theta, \mathbf{E}_p, \mathbf{H}) =: \psi_{ep}(\mathbf{E}_e, \theta, \mathbf{E}_p, \mathbf{H}) \quad (33)$$

and, consequently, for the derivatives (24) and (25)

$$\begin{aligned} \frac{\mathbf{T}}{\rho} &= \frac{\partial \psi_{ep}}{\partial \mathbf{E}_e}, \\ \eta &= -\frac{\partial \psi_{ep}}{\partial \theta}, \\ \frac{\partial \psi}{\partial \mathbf{E}_p} &= -\frac{\partial \psi_{ep}}{\partial \mathbf{E}_e} + \frac{\partial \psi_{ep}}{\partial \mathbf{E}_p}. \end{aligned} \quad (34)$$

The dissipation inequality (28) becomes herewith

$$\begin{aligned} \delta_m &= -\frac{\partial \psi}{\partial \mathbf{E}_p} \cdot \dot{\mathbf{E}}_p - \frac{\partial \psi}{\partial \mathbf{H}} \cdot \dot{\mathbf{H}} \\ &= \left(\frac{\mathbf{T}}{\rho} - \frac{\partial \psi_{ep}}{\partial \mathbf{E}_p} \right) \cdot \dot{\mathbf{E}}_p - \frac{\partial \psi_{ep}}{\partial \mathbf{H}} \cdot \dot{\mathbf{H}} \geq 0. \end{aligned} \quad (35)$$

The second term in the bracket can be identified as a back stress, so that the bracket itself represents what is called *effective stress*.

In analogy to Eq. (33), we can represent the entropy from Eq. (18) as

$$\eta = \eta(\mathbf{E}_e + \mathbf{E}_p, \theta, \mathbf{E}_p, \mathbf{H}) =: \eta_{ep}(\mathbf{E}_e, \theta, \mathbf{E}_p, \mathbf{H}). \quad (36)$$

By the definition of the free energy Eq. (12), we obtain for the internal energy

$$\varepsilon_{ep}(\mathbf{E}_e, \theta, \mathbf{E}_p, \mathbf{H}) := \psi_{ep}(\mathbf{E}_e, \theta, \mathbf{E}_p, \mathbf{H}) + \theta \eta_{ep}(\mathbf{E}_e, \theta, \mathbf{E}_p, \mathbf{H}). \quad (37)$$

Up to this point, our analysis has been rather general. In order to further specify this setting for plastic materials, we introduce another important assumption, which is the generalization of the assumption above of identical elastic behaviour.

In fact, for many materials like metals, it is by both physical reasons and experimental results well substantiated that the thermoelastic behaviour within the elastic ranges is almost constant under moderate plastic strains.³ We will therefore state in the following

Assumption *The thermoelastic behaviour within all elastic ranges of the elastoplastic material is identical.*

This means more precisely that in elastic events, all measurable quantities like the stresses, the heat flux, and the heat supply depend only upon the elastic strain and the temperature and their rates, but not on the plastic variables. The consequences of this assumption shall be exploited in the following.

By Eq. (35), we obtain for elastic events zero mechanical dissipation and hence from Eq. (14)

$$\delta_m = 0 \Rightarrow \mathbf{Q} = \theta \dot{\eta} \quad (38)$$

with the local heat supply on the left-hand side which we consider as a measurable quantity. With Eq. (36) the right-hand side becomes for elastic events

$$\theta \dot{\eta} = \theta \left(\frac{\partial \eta_{ep}}{\partial \mathbf{E}_e} \cdot \dot{\mathbf{E}}_e + \frac{\partial \eta_{ep}}{\partial \theta} \dot{\theta} \right), \quad (39)$$

which shall be independent of both \mathbf{E}_p and \mathbf{H} . This can only be the case iff the following decomposition of the entropy exists:

$$\eta = \eta_e(\mathbf{E}_e, \theta) + \eta_p(\mathbf{E}_p, \mathbf{H}). \quad (40)$$

After the above assumption, also the stresses shall not depend on the plastic variables. Hence, we have the reduced stress law

$$\mathbf{T} = T(\mathbf{E}_e, \theta) \quad (41)$$

instead of Eq. (24). By Eq. (34), we obtain

$$T(\mathbf{E}_e, \theta) = \rho \frac{\partial \psi_{ep}}{\partial \mathbf{E}_e} \quad (42)$$

and by Eqs. (34) and (40)

$$\eta_e(\mathbf{E}_e, \theta) + \eta_p(\mathbf{E}_p, \mathbf{H}) = - \frac{\partial \psi_{ep}}{\partial \theta}, \quad (43)$$

which leads to the split of the free energy

$$\psi = \psi_e(\mathbf{E}_e, \theta) - \theta \eta_p(\mathbf{E}_p, \mathbf{H}) + \varepsilon_p(\mathbf{E}_p, \mathbf{H}) \quad (44)$$

with

$$\psi_e(\mathbf{E}_e, \theta) := - \int \eta_e(\mathbf{E}_e, \theta) d\theta$$

³ In the context of finite plasticity, this assumption states the *isomorphy* of the thermoelastic laws, which is more complicated than in the geometrically linear theory at present, see [3,4].

or

$$\eta_e(\mathbf{E}_e, \theta) = -\frac{\partial \psi_e(\mathbf{E}_e, \theta)}{\partial \theta}$$

with an (up to now) arbitrary function ε_p of the plastic variables.

The internal energy follows from Eq. (37) as

$$\varepsilon = \varepsilon_e(\mathbf{E}_e, \theta) + \varepsilon_p(\mathbf{E}_p, \mathbf{H}) \quad \text{with} \quad \varepsilon_e(\mathbf{E}_e, \theta) := \psi_e(\mathbf{E}_e, \theta) + \theta \eta_e(\mathbf{E}_e, \theta). \quad (45)$$

The heat conduction can neither depend on \mathbf{E}_p and \mathbf{H} which gives rise to the reduced form of Eq. (18)

$$\mathbf{q} = q(\mathbf{E}_e, \theta, \mathbf{g}). \quad (46)$$

We state these findings in the following

Theorem *The assumption of equal thermoelastic behaviour in all elastic ranges is fulfilled if and only if the following representations hold for the*

- *entropy* $\eta = -\frac{\partial \psi_e(\mathbf{E}_e, \theta)}{\partial \theta} + \eta_p(\mathbf{E}_p, \mathbf{H})$
- *internal energy* $\varepsilon = \psi_e(\mathbf{E}_e, \theta) - \theta \frac{\partial \psi_e(\mathbf{E}_e, \theta)}{\partial \theta} + \varepsilon_p(\mathbf{E}_p, \mathbf{H})$
- *free energy* $\psi = \psi_e(\mathbf{E}_e, \theta) - \theta \eta_p(\mathbf{E}_p, \mathbf{H}) + \varepsilon_p(\mathbf{E}_p, \mathbf{H})$
- *heat conduction* $\mathbf{q} = q(\mathbf{E}_e, \theta, \mathbf{g})$
- *stresses* $\mathbf{T} = \rho \frac{\partial \psi_e(\mathbf{E}_e, \theta)}{\partial \mathbf{E}_e}$.

Such an additive split of the entropy and of the internal energy into purely thermoelastic and inelastic parts can already be found in the fundamental paper of Bridgman [7] who extended the principles of classical thermodynamics to one-dimensional plastic deformation and other irreversible phenomena.

Some statements concerning the uniqueness of these functions can be found in the last section.

Accordingly, the following functions completely constitute the thermoelastic material:

- the elastic free energy $\psi_e(\mathbf{E}_e, \theta)$
- the plastic entropy $\eta_p(\mathbf{E}_p, \mathbf{H})$
- the plastic internal energy $\varepsilon_p(\mathbf{E}_p, \mathbf{H})$
- the heat conduction law $q(\mathbf{E}_e, \theta, \mathbf{g})$
- the yield criterion $\varphi(\mathbf{E}, \theta, \mathbf{E}_p, \mathbf{H})$

together with the flow and hardening rules to be considered later.

Equation (44) gives for the residual dissipation inequality (35)

$$\begin{aligned} 0 &\leq \left(\mathbf{T} - \rho \frac{\partial \varepsilon_p}{\partial \mathbf{E}_p} + \rho \theta \frac{\partial \eta_p}{\partial \mathbf{E}_p} \right) \cdot \dot{\mathbf{E}}_p + \left(-\rho \frac{\partial \varepsilon_p}{\partial \mathbf{H}} + \rho \theta \frac{\partial \eta_p}{\partial \mathbf{H}} \right) \cdot \dot{\mathbf{H}} \\ &= \mathbf{T} \cdot \dot{\mathbf{E}}_p + \rho \theta \dot{\eta}_p(\mathbf{E}_p, \mathbf{H}) - \rho \dot{\varepsilon}_p(\mathbf{E}_p, \mathbf{H}). \end{aligned} \quad (47)$$

In order to determine the change of the temperature of the material point under consideration, we use the first law of thermodynamics (10) together with Eq. (45)

$$Q = \dot{\varepsilon} - \frac{\mathbf{T}}{\rho} \cdot \dot{\mathbf{E}} = \dot{\varepsilon}_e(\mathbf{E}_e, \theta) - \frac{\mathbf{T}}{\rho} \cdot \dot{\mathbf{E}}_e + \dot{\varepsilon}_p(\mathbf{E}_p, \mathbf{H}) - \frac{\mathbf{T}}{\rho} \cdot \dot{\mathbf{E}}_p. \quad (48)$$

This gives rise to the split of the heat supply

$$Q = Q_e + Q_p \quad (49)$$

into

$$Q_e(\dot{\mathbf{E}}_e, \theta, \dot{\mathbf{E}}_e, \dot{\theta}) := \dot{\varepsilon}_e(\mathbf{E}_e, \theta) - \frac{\mathbf{T}}{\rho} \cdot \dot{\mathbf{E}}_e = \theta \dot{\eta}_e(\dot{\mathbf{E}}_e, \theta) = -\theta \frac{\mathbf{M}}{\rho} \cdot \dot{\mathbf{E}}_e + c \dot{\theta} \quad (50)$$

by use of Eq. (38) with

- the *specific heat* $c(\mathbf{E}_e, \theta) := \theta \frac{\partial \eta_e}{\partial \theta}$
- the *stress-temperature tensor* $\mathbf{M}(\mathbf{E}_e, \theta) := -\rho \frac{\partial \eta_e}{\partial \mathbf{E}_e}$

and

$$Q_p := \dot{\varepsilon}_p(\mathbf{E}_p, \mathbf{H}) - \frac{\mathbf{T}}{\rho} \cdot \dot{\mathbf{E}}_p. \quad (51)$$

Equation (50) can be solved for

$$c \dot{\theta} = Q - Q_p + \theta \frac{\mathbf{M}}{\rho} \cdot \dot{\mathbf{E}}_e. \quad (52)$$

By this equation, we can integrate the temperature along the process and, thus, determine the final temperature after some elasto-plastic process. Its change is caused by

1. the heat supply Q from the outside
2. the heat $-Q_p$ generated by plastic yielding and hardening
3. thermoelastic transformations due to the last term in Eq. (52).

We now specify the ansatz for the rate-independent evolution Eq. (17) for the plastic variables decomposed into the *flow rule*

$$\dot{\mathbf{E}}_p = \lambda P(\mathbf{E}_e, \theta, \mathbf{E}_p, \mathbf{H}) \quad (53)$$

and the *hardening rule*

$$\dot{\mathbf{H}} = \lambda H(\mathbf{E}_e, \theta, \mathbf{E}_p, \mathbf{H}) \quad (54)$$

with a *plastic parameter* λ . It is zero if and only if no yielding occurs, that is, during elastic events. During yielding, however, it is positive. In all events, the Kuhn-Tucker-conditions hold in the form

$$\lambda \varphi = 0 \quad \text{with } \lambda \geq 0 \quad \text{and } \varphi \leq 0. \quad (55)$$

The plastic parameter can be determined during yielding by the *consistency condition* after Eq. (32)

$$\begin{aligned} 0 &= \dot{\varphi}(\mathbf{E}, \theta, \mathbf{E}_p, \mathbf{H}) \\ &= \frac{\partial \varphi}{\partial \mathbf{E}} \cdot \dot{\mathbf{E}} + \frac{\partial \varphi}{\partial \theta} \dot{\theta} + \frac{\partial \varphi}{\partial \mathbf{E}_p} \cdot \dot{\mathbf{E}}_p + \frac{\partial \varphi}{\partial \mathbf{H}} \cdot \dot{\mathbf{H}} \\ &= \frac{\partial \varphi}{\partial \mathbf{E}} \cdot \dot{\mathbf{E}} + \frac{\partial \varphi}{\partial \theta} \dot{\theta} + \frac{\partial \varphi}{\partial \mathbf{E}_p} \cdot \lambda P(\mathbf{E}_e, \theta, \mathbf{E}_p, \mathbf{H}) + \frac{\partial \varphi}{\partial \mathbf{H}} \cdot \lambda H(\mathbf{E}_e, \theta, \mathbf{E}_p, \mathbf{H}), \end{aligned} \quad (56)$$

which can be solved for the plastic parameter

$$\lambda = \alpha^{-1} \left(\frac{\partial \varphi}{\partial \mathbf{E}} \cdot \dot{\mathbf{E}} + \frac{\partial \varphi}{\partial \theta} \dot{\theta} \right) \quad (57)$$

with

$$\alpha := -\frac{\partial \varphi}{\partial \mathbf{E}_p} \cdot P(\mathbf{E}_e, \theta, \mathbf{E}_p, \mathbf{H}) - \frac{\partial \varphi}{\partial \mathbf{H}} \cdot H(\mathbf{E}_e, \theta, \mathbf{E}_p, \mathbf{H}). \quad (58)$$

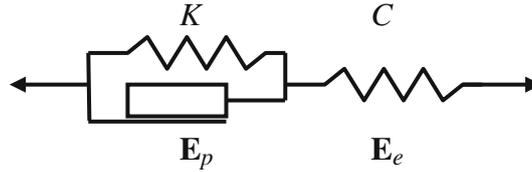
Due to the loading condition Eq. (21), $\lambda \alpha$ must be positive during plastic events. After Eq. (55), λ alone is positive, and so α must also be positive. After Eq. (58), this is a restriction on the functions φ , P , and H . Another restriction on these functions is obtained if we substitute Eqs. (53), (54) into the residual dissipation inequality (47)

$$\begin{aligned} &\left(\mathbf{T} - \rho \frac{\partial \varepsilon_p}{\partial \mathbf{E}_p} + \rho \theta \frac{\partial \eta_p}{\partial \mathbf{E}_p} \right) \cdot P(\mathbf{E}_e, \theta, \mathbf{E}_p, \mathbf{H}) \\ &+ \left(-\rho \frac{\partial \varepsilon_p}{\partial \mathbf{H}} + \rho \theta \frac{\partial \eta_p}{\partial \mathbf{H}} \right) \cdot H(\mathbf{E}_e, \theta, \mathbf{E}_p, \mathbf{H}) \geq 0. \end{aligned} \quad (59)$$

The consequences of this inequality are explained in the following section.

4 Example

In the following, we will particularize these results for a simple, but rather illustrative example, which can be derived from the following rheological model with two springs and an element of dry friction.



We first generalize this model into three dimensions. The strain of the spring C is \mathbf{E}_e , the strain of the spring K is \mathbf{E}_p , which is equal to that of the parallel friction element.

The elastic part of the free energy can be given the quadratic form familiar from thermoelasticity

$$\psi_e(\mathbf{E}_e, \theta) = \frac{1}{2\rho} \mathbf{E}_e \cdot \cdot \mathbf{C} \cdot \cdot \mathbf{E}_e + \frac{1}{\rho} \Delta\theta \mathbf{M} \cdot \cdot \mathbf{E}_e - \frac{c}{2\theta_0} \Delta\theta^2 \quad (60)$$

with a positive-definite elasticity tensor of fourth-order \mathbf{C} of the spring C , a constant symmetric stress temperature tensor \mathbf{M} , a constant specific heat c , a reference temperature θ_0 , and the deviation from it $\Delta\theta := \theta - \theta_0$. However, in order to guarantee that c represents indeed a constant specific heat according to Eqs. (44) and (50), this expression should be replaced by

$$\psi_e(\mathbf{E}_e, \theta) = \frac{1}{2\rho} \mathbf{E}_e \cdot \cdot \mathbf{C} \cdot \cdot \mathbf{E}_e + \frac{1}{\rho} \Delta\theta \mathbf{M} \cdot \cdot \mathbf{E}_e + c \left(\Delta\theta - \theta \ln \frac{\theta}{\theta_0} \right) \quad (61)$$

with negligible deviation from Eq. (60) in the case of small $|\Delta\theta|$ (see [27]).

We obtain with Eq. (42)

$$\mathbf{T} = \mathbf{C} \cdot \cdot \mathbf{E}_e + \Delta\theta \mathbf{M} \quad (62)$$

and with Eq. (44)

$$\eta_e(\mathbf{E}_e, \theta) = -\frac{1}{\rho} \mathbf{M} \cdot \cdot \mathbf{E}_e + c \ln \frac{\theta}{\theta_0} \quad (63)$$

and after Eq. (45)

$$\varepsilon_e(\mathbf{E}_e, \theta) = \frac{1}{2\rho} \mathbf{E}_e \cdot \cdot \mathbf{C} \cdot \cdot \mathbf{E}_e - \frac{1}{\rho} \theta_0 \mathbf{M} \cdot \cdot \mathbf{E}_e + c \Delta\theta. \quad (64)$$

Next we consider the plastic parts of the constitutive equations. The stresses in the spring K give the back stress

$$\mathbf{T}_B = \mathbf{K} \cdot \cdot \mathbf{E}_p \quad (65)$$

with a positive-semidefinite elasticity tensor of fourth-order \mathbf{K} . \mathbf{T}_B is assumed to be deviatoric, the same as \mathbf{E}_p .

We define the *specific plastic work* in some time interval $[t_0, t_1]$ as the work in the friction element in which the stress $\mathbf{T} - \mathbf{T}_B$ is acting:

$$w_p := \int_{t_0}^{t_1} \frac{\mathbf{T} - \mathbf{T}_B}{\rho} \cdot \cdot \dot{\mathbf{E}}_p dt \Rightarrow \dot{w}_p = \frac{\mathbf{T} - \mathbf{T}_B}{\rho} \cdot \cdot \dot{\mathbf{E}}_p. \quad (66)$$

It shall be chosen as a scalar hardening variable constituting \mathbf{H} .

If we use the Huber-v. Mises criterion (J_2 -theory)

$$\varphi(\mathbf{E}, \theta, \mathbf{E}_p, \mathbf{H}) = (\mathbf{T}' - \mathbf{T}_B) \cdot \cdot (\mathbf{T}' - \mathbf{T}_B) - \sigma_Y^2(w_p, \theta) \quad (67)$$

with \mathbf{T}' denoting the stress deviator, then the associated flow rule is

$$\dot{\mathbf{E}}_p = \lambda (\mathbf{T}' - \mathbf{T}_B). \quad (68)$$

The consistency condition (56) requires with Eqs. (62), (65), (66)

$$\begin{aligned} 0 = \dot{\varphi} &= 2 (\mathbf{T}' - \mathbf{T}_B) \cdot \cdot (\mathbf{C} \cdot \cdot \dot{\mathbf{E}}_e + \mathbf{M} \dot{\theta} - \mathbf{K} \cdot \cdot \dot{\mathbf{E}}_p) \\ &\quad - 2\sigma_Y \left(\frac{\partial \sigma_Y}{\partial w_p} \frac{\mathbf{T}' - \mathbf{T}_B}{\rho} \cdot \cdot \dot{\mathbf{E}}_p + \frac{\partial \sigma_Y}{\partial \theta} \dot{\theta} \right) \end{aligned} \quad (69)$$

or

$$\begin{aligned} &(\mathbf{T}' - \mathbf{T}_B) \cdot \cdot (\mathbf{C} \cdot \cdot \dot{\mathbf{E}} + \mathbf{M} \dot{\theta}) - \sigma_Y \frac{\partial \sigma_Y}{\partial \theta} \dot{\theta} \\ &= (\mathbf{T}' - \mathbf{T}_B) \cdot \cdot \left(\mathbf{C} + \mathbf{K} + \sigma_Y \frac{\partial \sigma_Y}{\rho \partial w_p} \mathbf{I} \right) \cdot \cdot \dot{\mathbf{E}}_p \\ &= \lambda (\mathbf{T}' - \mathbf{T}_B) \cdot \cdot \left(\mathbf{C} + \mathbf{K} + \sigma_Y \frac{\partial \sigma_Y}{\rho \partial w_p} \mathbf{I} \right) \cdot \cdot (\mathbf{T}' - \mathbf{T}_B) \end{aligned} \quad (70)$$

with the fourth-order identity \mathbf{I} . In case of yielding, the left-hand side is positive due to the loading condition. If there is hardening

$$\frac{\partial \sigma_Y}{\partial w_p} \geq 0, \quad (71)$$

then the fourth-order tensor on the right-hand side is positive definite so that the condition $\alpha > 0$, which we stated after Eq. (58), is satisfied. Note that Eqs. (66) and (68) imply that the plastic power and, hence, also the plastic work are positive:

$$\dot{w}_p > 0 \Rightarrow w_p > 0. \quad (72)$$

This is a consequence of the loading condition and has nothing to do with the dissipation inequality.

This plastic work is only partly converted into heat. The other part might go into structural changes such as phase transitions, (de)bonding, etc. After the classical experiments by Farren and Taylor [11] and Taylor and Quinney [29] the part turned into heat lies typically in the range of 85–95 % for metals. It can be implemented into our format by a Taylor–Quinney-factor β , which takes values between 0 and 1, and is (here) assumed to be constant. On the other hand, strain energy will also be stored in the spring K . So we make the following ansatz for the plastic part of the internal energy:

$$\varepsilon_p(\mathbf{E}_p, w_p) = \frac{1}{2\rho} \mathbf{E}_p \cdot \cdot \mathbf{K} \cdot \cdot \mathbf{E}_p + (1 - \beta) w_p. \quad (73)$$

The heat generated by yielding is then according to Eqs. (51), (65), and (72) indeed

$$\begin{aligned} -Q_p &= \frac{\mathbf{T}}{\rho} \cdot \cdot \dot{\mathbf{E}}_p - \dot{\varepsilon}_p(\mathbf{E}_p, w_p) \\ &= \frac{\mathbf{T} - \mathbf{T}_B}{\rho} \cdot \cdot \dot{\mathbf{E}}_p - (1 - \beta) \dot{w}_p = \beta \dot{w}_p > 0. \end{aligned} \quad (74)$$

If the material element is adiabatically insulated, then we obtain a change of temperature and, eventually, a thermal expansion after Eq. (52).

The dissipation inequality (47) becomes with Eqs. (51) and (74)

$$-Q_p + \theta \dot{\eta}_p = \beta \dot{w}_p + \theta \dot{\eta}_p \geq 0 \quad (75)$$

and imposes a restriction on the plastic part of the entropy. The trivial assumption $\eta_p = 0$ obviously satisfies this inequality. However,

$$\eta_p(w_p) = \gamma w_p \quad (76)$$

with some positive real constant γ would also be an admissible choice since it fulfils the dissipation inequality in the form

$$(\beta + \gamma\theta) \dot{w}_p \geq 0. \quad (77)$$

For the plastic part of the entropy, no further concretizations can be made on the basis of measurements.

The fact that the plastic part of the entropy will generally not be unique can already be inferred from the inequality (47). We may augment η by any increasing function like, e.g., any monotonous function of an increasing hardening parameter.

With Eq. (44) we obtain for the free energy

$$\psi = \frac{1}{2\rho} \mathbf{E}_e \cdot \cdot \mathbf{C} \cdot \cdot \mathbf{E}_e + \frac{1}{\rho} \Delta\theta \mathbf{M} \cdot \cdot \mathbf{E}_e + c \left(\Delta\theta - \theta \ln \frac{\theta}{\theta_0} \right) + \frac{1}{2\rho} \mathbf{E}_p \cdot \cdot \mathbf{K} \cdot \cdot \mathbf{E}_p + (1 - \beta - \theta\gamma) w_p. \quad (78)$$

If we add a Fourier-type heat flux equation, then the material model is completed.

In our model *yielding against the stresses* can take place, that is, $\mathbf{T} \cdot \cdot \dot{\mathbf{E}}_p$ can be negative without contradicting the dissipation inequality. In many papers, the positive semi-definiteness of this term is taken as a sufficient condition for the dissipation inequality to hold. However, we have seen that it is neither necessary nor sufficient. Moreover, there are clear experimental findings that in real materials, the Bauschinger effect can be so strong that this effect takes place, see [8, 15, 24, 25].

5 Conclusions

It is demonstrated how one can enlarge the classical theory of plasticity by thermodynamic variables in a natural way. The key feature is the concept of *identical thermoelastic behaviour within the thermoelastic ranges*. As a result of this assumption, we obtain certain necessary and sufficient conditions, among them a split of the internal energy, the entropy, and the free energy into elastic and plastic parts.

The second law is applied in the form of the Clausius–Duhem-inequality. This gives certain restrictions of the constitutive equations, in particular, concerning the flow and hardening rules. The positive semi-definiteness of $\mathbf{T} \cdot \cdot \dot{\mathbf{E}}_p$ turns out to be neither sufficient nor necessary for the second law. This has been demonstrated in a simple example of a material with kinematic hardening based on a rheological model.

It has also been shown how the temperature change due to plastic heat production and other sources can be determined after introduction of a Taylor–Quinney-factor.

Since we assumed that the stresses \mathbf{T} and the heat supply Q_e during elastic events are accessible to measurement, the rates of the elastic parts of the internal energy and the entropy can be inferred from Eq. (50). However, ε_e and η_e are only unique up to additive constants which cannot be determined by measurements and, thus, have no influence on the thermomechanical behaviour. If we replace ε_e by $\varepsilon_e + c_1$ and η_e by $\eta_e + c_2$ then the elastic part of the free energy has to be replaced by $\psi_e + c_1 - c_2\theta$ according to Eq. (45) and is thus seen to be unique only up to a linear function of the temperature.

Since the additional heat supply Q_p during plastic events is also accessible to measurement, the rate of the plastic part of the internal energy is determined by Eq. (51), but ε_p is unique only up to an additive constant.

After all, the plastic part of the entropy is only weakly restricted by the dissipation inequality (47). Our example has shown that different functions of the plastic variables satisfy this inequality, and none of them can be distinguished on the basis of measurements. This implies that the plastic part of the free energy is also by no means unique. It must be guaranteed that at least one expression of η_p can be found that satisfies the inequality. Such an expression, however, is not required for any prediction. This vagueness of the entropy in the context of plasticity was already pointed out by Coleman and Owen [10].

References

1. Acharya, A., Shawki, T.G.: The Clausius–Duhem inequality and the structure of rate-independent plasticity. *Int. J. Plast.* **22**(2), 229–283 (1996)
2. Bergander, H., Luther, M.: Zur Dissipation beim plastischen Fließen. *Tech. Mech.* **6**(2), 58–65 (1985)
3. Bertram, A.: An alternative approach to finite plasticity based on material isomorphisms. *Int. J. Plast.* **15**(3), 353–374 (1999)
4. Bertram, A.: *Elasticity and Plasticity of Large Deformations—An Introduction*. Springer, Berlin (2005, 2008, 2012)

5. Bever, M.G., Hold, D.L., Tichener, A.L.: The stored energy of cold work. *Prog. Mater. Sci.* **17**, 1–190 (1973)
6. Böhlke, T., Bertram, A.: The evolution of Hooke's law due to texture development in polycrystals. *Int. J. Solids Struct.* **38**(52), 9459–9470 (2001)
7. Bridgman, P.W.: The thermodynamics of plastic deformation and generalized entropy. *Rev. Mod. Phys.* **22**(1), 56–63 (1950)
8. Brown, A.A., Casey, J., Nikkel, D.J.: Experiments conducted in the context of the strain-space formulation of plasticity. *Int. J. Plast.* **19**(11), 1965–2005 (2003)
9. Casey, J.: On elastic-thermo-plastic materials at finite deformations. *Int. J. Plast.* **14**(1–3), 173–191 (1998)
10. Coleman, B.D., Owen, D.R.: On thermodynamics and elastic-plastic materials. *Arch. Ration. Mech. Anal.* **59**, 25–51 (1975)
11. Farren, W.S., Taylor, G.I.: The heat developed during plastic extension of metals. *Proc. R. Soc. Lon. A* **107**, 422–451 (1925)
12. Germain, P., Nguyen, Q.S., Suquet, P.: Continuum thermodynamics. *J. Appl. Mech.* **50**, 1010–1020 (1983)
13. Haupt, P.: Thermodynamics of solids. In: Muschik, W. (eds.) *Non-Equilibrium Thermodynamics with Application to Solids*, CISM Course, vol. 336. Springer, Wien (1993)
14. Houlsby, G.T., Puzrin, A.M.: A thermomechanical framework for constitutive models for rate-independent dissipative materials. *Int. J. Plast.* **16**, 1017–1047 (2000)
15. Ikegami, K.: Experimental plasticity on the anisotropy of metals. In: *Colloques internationaux du CNRS 295 Comportement mécanique de solides anisotropes*, pp. 201–242 (1982)
16. Kamlah, M., Haupt, P.: On the macroscopic description of stored energy and self heating during plastic deformation. *Int. J. Plast.* **13**(10), 893–911 (1998)
17. Krawietz, A.: *Materialtheorie*. Springer, Berlin (1986)
18. Lehmann, T. (ed.): *The Constitutive Law in Thermoplasticity*. CISM Course 281. Springer, Wien (1984)
19. Lin, R.C., Brocks, W., Betten, J.: On dissipation inequalities and finite strain inelastic constitutive laws: theoretical and numerical comparisons. *Int. J. Plast.* **22**, 1825–1857 (2006)
20. Lubliner, J.: On the thermodynamic foundations of non-linear solid mechanics. *Int. J. Non Linear Mech.* **7**, 237–254 (1972)
21. Lucchesi, M., Silhavy, M.: Thermoplastic materials with combined hardening. *Int. J. Plast.* **9**, 291–315 (1993)
22. Maugin, G.A.: *The Thermomechanics of Plasticity and Fracture*. Cambridge University Press, Cambridge (1992)
23. Maugin, G. A.: *Thermomechanics of Nonlinear Irreversible Behaviors—An Introduction*. World Sci. Singapore (1999)
24. Phillips, A., Liu, C.S., Justusson, J.W.: An experimental investigation of yield surfaces at elevated temperatures. *Acta Mech.* **14**, 119–146 (1972)
25. Phillips, A.: The foundations of thermoplasticity—experiments and theory. In: Zeman, J.L., Ziegler, F. (eds.) *Topics in Applied Continuum Mechanics*, Springer-Verlag, Wien, pp. 1–21 (1974)
26. Ristinmaa, M., Wallin, M., Ottosen, N.S.: Thermodynamic format and heat generation of isotropic hardening plasticity. *Acta Mech.* **194**, 103–121 (2007)
27. Rosakis, P., Rosakis, A.J., Ravichandran, G., Hodowany, J.: A thermodynamic internal variable model for the partition of plastic work into heat and stored energy in metals. *J. Mech. Phys. Solids* **48**, 581–607 (2000)
28. Svendsen, B.: A thermodynamic formulation of finite deformation elastoplasticity with hardening based on the concept of material isomorphism. *Int. J. Plast.* **14**(6), 473–488 (1998)
29. Taylor, G.I., Quinney, H.: The latent heat remaining in a metal after cold working. *Proc. R. Soc. Lon. A* **163**, 157–181 (1937)