Stability analysis and simulations of nonlinear rotor-shaft joints with internal damping

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Abstract

Within this contribution analytical and numerical models are stated to describe dissipation processes, which appear if rubber elements serve as rotor shaft joint in rotor systems. The effects of internal damping in combination with nonlinear stiffness in joints are worked out. It is shown that the nonlinearity has a positive effect on the stability range. Measurements on a basic rotor test rig which are depicted in the paper approve the analytical and numerical results.

1 Introduction

In some cases the use of elastomer as joint between rotor and shaft is expedient in the design of lab centrifuges. These materials are utilized to facilitate the change of rotors and to uncouple rotor vibrations from shaft movements. They allow a self centering of the rotor with sufficiently low bearing loads even with high speeds and high unbalance loads. However by the use of rubber elements viscous internal damping occurs in the rotating system leading to instabilities in the post critical speed range.

On a lab centrifuge with an o-ring embedded between rotor and shaft large rotor deflections have been observed above a certain speed of revolution [4]. The increased amplitudes could be traced back to internal damping working in the o-ring. Due to the clearance between rotor and shaft the restoring force gets nonlinear already with small relative movements. The nonlinear stiffness of the o-ring is one reason which effects, that the amplitudes stay bounded.

One focus of resent research in internal damping of rotor systems is the development of more accurate linear and nonlinear simulation models and simulation techniques in order to avoid instabilities and to get a more in-depth understanding of the arising mechanisms [6],[2],[1].

In this paper an analytical nonlinear four degrees of freedom rotor system model is stated. The nonlinear rotor shaft joint is modeled with a spring damper element. Considering the spring, a force law is used, where the spring force depends cubic on the deflection. If an o-ring connects rotor and shaft the nonlinear o-ring properties can be described by this model in a clear manner.

In [5] a cubic force law for the nonlinear restoring force of a rotor model with two degrees of freedom is applied, too. It is shown, that the nonlinearity causes an enlargement of the stable frequency range towards higher speeds of revolution. The model with two degrees of freedom describes basic relations very well. Especially this is the case, if the nonlinearity and the internal damping take effect mostly in the shaft of the rotor system.

With the subsequently proposed rotor system model with four degrees of freedom it is possible to analyze the effects of nonlinear restoring forces within the rotor shaft joint and to work out the influence of each single parameter, like shaft stiffness, joint stiffness, mass of rotor and mass of shaft.

The steady state solution in the unbalance case and the stability bounds are calculated analytically. They show a good agreement with numerical results, which are obtained with Feran, a rotor dynamic simulation tool developed by the authors [3].

A model, representing a rotor-shaft connection with an o-ring has been analyzed by a hydro pulse rig. Beneath the linear way, two further approaches to describe the measured hysteresis, a cubic and a bilinear cubic force law are shown in the paper. The different analytical and numerical results for the whole rotor system with these three approaches are compared.

The analysis is completed by measurements on a basic rotor test rig.
2 Analytical solution and stability bounds of a rotor system with four degrees of freedom and nonlinear rotor shaft joint

In figure 1 a model of a Jeffcott rotor with four degrees of freedoms is shown. The center of gravity of the shaft with mass \( m_u \) is deflected by \( r_1 \). The shaft stiffness is denoted by \( k_w \), the mass of the rotor by \( m_R \). The rotor shaft joint consists of a viscous internal damper with inner damping coefficient \( c_i \) and of a nonlinear spring. The nonlinear force law

\[
F_o = k_i \cdot \Delta r + k_{int} \cdot \Delta r^3
\]

is used to describe the spring, with \( k_i \), the linear stiffness of the rotor shaft joint, \( k_{int} \) the nonlinear stiffness of the rotor shaft joint and \( \Delta r \), the relative displacement between shaft and rotor. The coefficient of the outer damping is \( c_o \). The rotor system is subject to an unbalance load with mass of unbalance \( m_u \) and radial distance between axis of rotation and unbalance load \( r_u \).

In order to calculate the steady state solution it is advantageous to formulate the equation of motion in polar coordinates. The deflection of the shaft is described by the coordinates \( r_1 \) and \( \varphi_1 \) and the one of the rotor by \( r_2 \) and \( \varphi_2 \). In a further step the stability of the steady state case is analyzed. The equation of motion in a complex manner is:

\[

t = \frac{\Delta r}{2}, \quad \varphi = \frac{\Delta \varphi}{2}, \quad \Delta r \approx \Delta \varphi = \Delta r = \Delta \varphi = 0
\]

Furthermore it is \( \varphi_1 = 0 \) and \( \varphi_2 = 0 \). Hence \( \varphi_1 \) and \( \varphi_2 \) are constant in respect of time. With \( \alpha = \varphi_1 \) and \( \beta = \varphi_2 \) one gets via integration \( \varphi_1 = \alpha t + C_1 \) and \( \varphi_2 = \beta t + C_2 \). As the rotor could not twist unbounded in respect to the shaft, \( \varphi_2 - \varphi_1 = (\beta - \alpha)t + C_2 - C_1 \) is bounded. It follows that \( \beta - \alpha = 0 \) and hence \( \varphi_1 = \varphi_2 = \varphi \).

The following relative coordinates are introduced: \( \varphi_0 = \varphi_1 = \Delta \varphi_0 = \varphi_2 = \Delta \varphi_2 = \Delta \varphi \). The subcript \( 0 \) denotes that the steady state case is considered. Furthermore \( |\Delta \varphi_0| \ll 1 \) and hence \( \cos \Delta \varphi_0 \approx 1 \) and \( \sin \Delta \varphi_0 \approx \Delta \varphi_0 \). This yields \( e^{i(\varphi_0 - \varphi_0)} = 1 + i \Delta \varphi_0 \) and \( e^{i(\varphi_0 - \varphi_0)} = 1 - i \Delta \varphi_0 \). One gets four real and coupled differential equations:

\[
-m_w(\dot{r}_1^2 r_1 + (c_i + c_u)\dot{r}_1 - c_i \dot{r}_2 e^{i(\varphi_2 - \varphi_1)} + (k_w + \tilde{k})r_1 - \tilde{k} r_2 e^{i(\varphi_2 - \varphi_1)} + i[m_w(2\dot{r}_1 + \dot{\varphi}_1) + (c_i + c_u)\varphi_1] + c_i \dot{r}_2 r_2 e^{i(\varphi_2 - \varphi_1)} + (\Omega \varphi_1 r_1 + \Omega c_i r_2 e^{i(\varphi_2 - \varphi_1)}) = 0
\]

\[
m_R(\dot{r}_2 - \dot{\varphi}_2) + c_i \dot{r}_2 - c_i \dot{r}_1 e^{i(\varphi_2 - \varphi_1)} + \tilde{k} r_2 e^{i(\varphi_2 - \varphi_1)} + i[m_R(2\dot{r}_2 + \dot{\varphi}_2) + (c_i + c_u)\varphi_2 r_2 - c_i \dot{r}_1 r_1 e^{i(\varphi_1 - \varphi_2)} - \Omega c_i r_2 e^{i(\varphi_2 - \varphi_1)}]
\]

\[
= m_u r_a \Omega^2 e^{i(\Omega t - \varphi_0)}
\]

with \( \tilde{k} = k_i + k_{int} \cdot |\Delta r|^2 \). In the steady state case all time derivatives of the coordinates \( r_1 \) and \( r_2 \) vanish. Furthermore it is \( \dot{r}_1 = 0 \) and \( \dot{r}_2 = 0 \). Hence \( \dot{\varphi}_1 \) and \( \dot{\varphi}_2 \) are constant in respect of time. With \( \alpha = \varphi_1 \) and \( \beta = \varphi_2 \) one gets via integration \( \varphi_1 = \alpha t + C_1 \) and \( \varphi_2 = \beta t + C_2 \). As the rotor could not twist unbounded in respect to the shaft, \( \varphi_2 - \varphi_1 = (\beta - \alpha)t + C_2 - C_1 \) is bounded. It follows that \( \beta - \alpha = 0 \) and hence \( \varphi_1 = \varphi_2 = \varphi \).

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\[
-m_w(\varphi_0^2 r_1 + k_w r_1 - (k_i + k_{int})|\Delta r_0|^2) r_0 + c_i (\varphi_0 - \Omega) r_20 \Delta \varphi_0 = 0
\]

\[
-m_R\varphi_0^2 r_20 + (k_i + k_{int})|\Delta r_0|^2) r_0 - c_i (\varphi_0 - \Omega) r_10 \Delta \varphi_0 = m_u r_a \Omega^2 \cos(\Omega t - \varphi_20)
\]

\[
c_i \varphi_0 r_10 - c_i \varphi_0 \Delta r_0 + \Omega c_i \Delta r_0 + (k_i + k_{int})|\Delta r_0|^2) r_10 \Delta \varphi_0 = 0
\]

\[
c_i \varphi_0 \Delta r_0 - \Omega c_i \Delta r_0 + (k_i + k_{int})|\Delta r_0|^2) r_10 \Delta \varphi_0 = m_u r_a \Omega^2 \sin(\Omega t - \varphi_20)
\]
In order to allow a further analytical calculation of the steady state solution with a limited effort, the case without outer damping is analyzed.

With equations (5) and (6) one obtains with \( c_a = 0 \) and \( \ddot{\varphi}_{20} = 0 \) the condition \( \ddot{\varphi}_0 = \Omega \) and finally it follows with \( \Delta r_0 \neq 0 \) that \( \Delta \varphi_0 = 0 \). With (3) and (4) one gets:

\[
a \Delta r_0^2 + b \Delta r_0 - c = 0 \tag{7}
\]

with

\[
a = a(\Omega) = (\frac{m_w}{m_R} + \frac{k_w}{m_R \Omega^2} - 1)k_{inl}
\]

\[
b = b(\Omega) = -\frac{m_w}{m_R} k_i + m_w \Omega^2 + \frac{k_w k_i}{m_R \Omega^2} - k_w - k_i
\]

\[
c = c(\Omega) = (k_w - m_w \Omega^2) \frac{m_w}{m_R} r_u
\]

Factor \( a \) contains the nonlinear stiffness, factor \( b \) only linear properties and factor \( c \) the unbalance load. With polynomial (7) the steady state solution \( \Delta r_0 \) can be calculated.

### 2.1 Stability of the steady state solution

The stability of the steady state solution \( \Delta r_0 \) is analyzed by stating the equation of motion (2) in cartesian coordinates and subsequently by a linearization at \( \Delta r_0 \). The generalized coordinates of the rotor system are \( \{x\} = \{x_1, x_2, y_1, y_2\}^T \). The linearized equation of motion written in a general formulation is:

\[
[M] \cdot \{x\}'' + [C] \cdot \{x\}' + [[K] + [P]] \cdot \{x\} = \{0\} \tag{8}
\]

with mass matrix \( [M] \), damping matrix \( [C] \), stiffness matrix \( [K] \) and matrix \( [P] \) consisting of parts which are a result of internal damping.

In [3] the stability of a linear rotor system with four degrees of freedom was analyzed with the help of the theorem of Thomson and Tait. In this case the equation of motion has to be transformed in an appropriate way. The same approach can be adopted on the linearized equation of motion (8). One obtains a stiffness matrix \( [K^*] \) which has to be positive definite in order to get an asymptotically stable system. If \( [K^*] \) is semi definite, the system is stable but not asymptotically stable ([7]). This condition of stability allows to deduce three stability functions \( s_1(\Omega, \Delta r_0) \), \( s_2(\Omega, \Delta r_0) \) and \( s_3(\Omega, \Delta r_0) \). If they are positive, the matrix \( [K^*] \) is positive definite and the steady state solution is stable.

\[
s_1(\Omega, \Delta r_0) = -\frac{m_w}{k_i + 3k_{inl} \Delta r_0^2} \Omega^2 + 1 + \frac{k_w}{k_i + 3k_{inl} \Delta r_0^2} \tag{9}
\]

\[
s_2(\Omega, \Delta r_0) = 1 - \frac{m_R}{k_i + 3k_{inl} \Delta r_0^2} \Omega^2 \tag{10}
\]

\[
s_3(\Omega, \Delta r_0) = -\frac{m_w m_R}{(k_i + 3k_{inl} \Delta r_0^2)^2} \Omega^4 - \left(\frac{m_w}{m_R} + 1 + \frac{k_w}{k_i + 3k_{inl} \Delta r_0^2}\right) \frac{m_R}{k_i + 3k_{inl} \Delta r_0^2} \Omega^2 + \frac{k_w}{k_i + 3k_{inl} \Delta r_0^2} \tag{11}
\]

### 2.2 Steady state solution and stability of the rotor system without unbalance load

If no unbalance load is existent, \( c = 0 \) in equation (7) and

\[
(a \Delta r_0^2 + b) \Delta r_0 = 0. \tag{12}
\]
Equation (12) has two solutions, the trivial solution $\Delta r_0 = 0$ and the nontrivial solution

$$\Delta r_0 = \sqrt{-\frac{b}{a}}. \quad (13)$$

By this two solutions two characteristic frequencies can be deduced. The condition for existence of a nontrivial solution is $\frac{b}{a} < 0$ for $a \neq 0$. The frequency of revolution, above which nontrivial solutions appear is computed by the condition that $b = 0$. It is fulfilled for frequencies of revolution

$$\Omega_1^{2} = \frac{k_i}{2m_R} + \frac{k_w + k_i}{2m_w} \mp \sqrt{\frac{k_i}{2m_R} + \frac{k_w + k_i}{2m_w}} - \frac{k_w k_i}{m_R m_w}. \quad (14)$$

The frequency $\Omega_1$ corresponds to the boundary frequency of the linear system with internal damping respectively to the lower resonance frequency of the linear system which is analyzed in [3]. The boundary frequency $\Omega_1$ separates stable from unstable speed range of the linear system. The root $a(\Omega_w) = 0$ is

$$\Omega_a = \sqrt{\frac{k_w}{m_w + m_R}}. \quad (15)$$

The nontrivial solution of $\Delta r_0$ in equation (12) exists on the condition that $b > 0$ and $a < 0$ or $b < 0$ and $a > 0$. It occurs between the frequencies $\Omega_1$ and $\Omega_a$ and if the frequency of revolution is higher than $\Omega_2$. It is possible to proof that $\Omega_1 < \Omega_a < \Omega_2$.

In figure 2 the positions of the characteristic frequencies $\Omega_1, \Omega_a$ and $\Omega_2$ are localized. The displacement $\Delta r_0$ is shown which is obtained from equation (12). Frequency ranges where the trivial solution $\Delta r_0 = 0$ appears can be separated clearly from frequency ranges with a nontrivial solution $\Delta r_0 = \sqrt{-b/a}$.

The steady state case is asymptotically stable if the functions $s_1$, $s_2$ and $s_3$ are positive ((9), (10), (11)). For

![Figure 2](image-url)

**Figure 2:** Displacement of rotor $r_{20}$, shaft $r_{10}$ and relative displacement $\Delta r_0$ for a nonlinear rotor system without unbalance load (left). Stability functions $s_1$, $s_2$ and $s_3$ of the rotor system (right). Ratio $\chi = \frac{\Omega_a}{\Omega_1} = 1.26$. Parameters for calculation are $m_w = 0.779 \text{ kg}$, $m_R = 4 \text{ kg}$, $k_w = 2.15 \cdot 10^6 \text{ N/m}$, $k_i = 2.7 \cdot 10^6 \text{ N/m}$, $k_{int} = 2.03 \cdot 10^{16} \text{ N/m}^3$, $c_i = 1.54 \cdot 10^5 \text{ Ns/m}$.

frequencies of revolution $0 \leq \Omega < \Omega_1$ the three stability functions pass with the solution $\Delta r_0 = 0$ onto the stability conditions of the linear system. As the linear system is stable in this frequency range the stability functions $s_1$, $s_2$, $s_3$ are positive for $0 \leq \Omega < \Omega_1$ (13]). The solution $\Delta r_0 = 0$ is unstable above $\Omega_1$. Hence the steady state solution should change into the nontrivial solution $\Delta r_0 = \sqrt{-b/a}$ whose stability is analyzed in the following.
The stability condition \( s_1 > 0 \) (9) is fulfilled at all frequencies \( \Omega < \Omega_{s1} = \sqrt{(k_i + k_w)/m_w} \). Furthermore it holds that \( \Omega_a < \Omega_{s1} \). Thus function \( s_1 \) is positive in the range between \( \Omega_1 \leq \Omega \leq \Omega_a \). By an extensive proof one can show that \( s_2 \) is positive in this frequency range, too. Function \( s_3 \) is positive for all frequencies of revolution \( 0 \leq \Omega \leq \Omega_1 \). For \( \Omega_1 \leq \Omega < \Omega_a \) one obtains \( s_3 = 0 \), if one substitute the solution \( \Delta r_0 = -b/a \) into \( s_3 \). The system is stable in this speed range but not asymptotically stable. Above the frequency \( \Omega_a \) the function \( s_3 \) becomes negative.

In figure 2 the functions \( s_1 \), \( s_2 \) and \( s_3 \) are depicted. Above the stability bound \( \Omega_a \) at least one of the functions becomes negative. In this case function \( s_3 \) is the first one.

In summary it was shown for the case without unbalance load that the stability bound of the nonlinear system is \( \Omega_a = \sqrt{\frac{k_w}{m_w + m_R}} \). The rotor system is asymptotically stable up to the linear boundary frequency \( \Omega_1 \) with \( \Delta r_0 = 0 \). Above the linear boundary frequency up to the stability bound \( \Omega_a \) the system has the steady state solution \( \Delta r_0 = \sqrt{-\frac{b}{a}} \) which is stable with \( s_3 = 0 \) but not asymptotically stable.

The nonlinear stiffness of the rotor shaft joint \( k_{inl} \) has no influence to the stability bound \( \Omega_a \). However \( k_{inl} \) influence the amplitudes which occur between the linear boundary frequency \( \Omega_1 \) and the stability bound \( \Omega_a \). The higher the nonlinear stiffness \( k_{inl} \) is the lower are the relative deflections \( \Delta r_0 \). The limit of the steady state solution for an infinitely high nonlinear joint stiffness is \( \lim_{k_{inl} \to \infty} (\Delta r_0) = \lim_{k_{inl} \to \infty} (-b/a) = 0 \). Even though the rotor system is stable beneath \( \Omega_a \) the amplitudes of the rotor could get very large between \( \Omega_1 \) and \( \Omega_a \). Especially in the vicinity of \( \Omega_a \) large amplitudes can be expected leading to loads which could not be supported by the rotor system. The nonlinear stiffness influences decisively the speed up to which the amplitudes are low enough to guaranty a safe operation of the rotor system.

### 2.2.1 Influence of the stiffness to the stability of the rotor system without unbalance

The ratio \( \chi \) of linear and nonlinear stability margin is defined as \( \chi = \frac{\Omega_a}{\Omega_1} \). In figure 3 \( \chi \) is depicted for different stiffness and mass ratios \( \kappa = \frac{k_w}{k_1} \) and \( \rho = \frac{m_w}{m_R} \).

![Figure 3: Ratio \( \chi \) of linear and nonlinear stability margin for different stiffness and mass ratios \( \kappa = \frac{k_w}{k_1} \) and \( \rho = \frac{m_w}{m_R} \).](image)

For a stiff shaft and a weak joint \( \kappa = \frac{k_w}{k_1} \) gets very high. For high values of \( \kappa \) the boundary frequency of the linear four degrees of freedom model approaches \( \Omega_1 = \sqrt{\frac{k_1}{m_w + m_R}} \). The nonlinear stability bound tends to infinity. Thus

\[
\lim_{\kappa \to \infty} \left( \frac{\Omega_a}{\Omega_1} \right) = \lim_{\kappa \to \infty} \left( \sqrt{\frac{k_w m_R}{k_1 (m_w + m_R)}} \right) = \lim_{\kappa \to \infty} \left( \sqrt{\frac{\kappa}{\rho + 1}} \right) \to \infty
\] (16)
With a stiff shaft and a weak connection to the rotor the stability bound of the linear Jeffcott rotor is mainly influenced by joint stiffness and mass of rotor. Depending on the nonlinearity, the mass ratio $\rho$ has now a large influence on the stability whereas a large shaft mass has a negative impact to the enlargement of the stability area by the nonlinearity (figure 3).

The linear boundary frequency $\Omega_1$ is obtained from the condition $b(\Omega_1) = 0$ (13). This condition is used to compute the limit of the linear boundary frequency $\Omega_1$ for a very stiff rotor shaft joint:

$$\lim_{k_i \to \infty} (\Omega_1) = \sqrt{\frac{k_w}{m_w + m_R}}$$  \hspace{1cm} (17)

In this case the boundary frequency $\Omega_a$ equals approximately the resonance frequency of the linear rotor system $\Omega_1$. Hence the nonlinearity has nearly no impact to the stability of the rotor system if the rotor shaft joint is very stiff.

### 2.3 Steady state solution and stability of the nonlinear rotor system with unbalance load

If a rotor system with unbalance load is considered equation (7) leads to the steady state solution:

$$\Delta r_0 = \frac{\sqrt{3}}{6\alpha} - \frac{2b}{\sqrt{3}\alpha} \hspace{1cm} \alpha = (108c + 12\sqrt{3} \sqrt{4b^3 + 27c^2a})a^2.$$  \hspace{1cm} (18)

Two further solutions are complex. Therefore they are not considered more closely. For some values the radicand $\frac{4b^3 + 27c^2a}{a}$ becomes negative during the calculation of $\alpha$. In this case one obtains from an extensive calculation the steady state solution:

$$\Delta r_0 = \sqrt{-\frac{4b}{3a} \cos \left(\frac{1}{3} \arccos \left(-\frac{27c^2a}{4b^3}\right)\right)}$$  \hspace{1cm} (19)

If there is no nonlinearity, equation (7) yields the steady state solution $\Delta r_0 = \frac{c}{b}$. Here $\lim_{\Omega \to \infty} \left(\frac{c}{b}\right) = \lim_{\Omega^2 \to \infty} \left(\frac{c\Omega^2}{b\Omega^2}\right) = -\frac{m_u}{m_R} r_u$ holds.

The stability margin of the system linearized at $\Delta r_0$ follows from the stability functions $s_1$, $s_2$ and $s_3$ ((9),(10),(11)).

The singular point $a = 0$ in solutions (18) and (19) occurs independently of the unbalance load $m_u$ for

$$\Omega_a = \sqrt{\frac{k_w}{m_w + m_R}}.$$  \hspace{1cm} (20)

It can be shown that the singularity $a = 0$ is located beyond the stability bound $\Omega_1$ of the linear system. At a frequency of revolution $\Omega = \Omega_a$ this system corresponds with $a = 0$ to a linear system with the steady state solution $\Delta r_0 = \frac{c}{b}$ (7). This solution would be unstable, as the boundary frequency of the linear system is smaller than the one of the nonlinear system.

In figure 4 the steady state solution $\Delta r_0$ is depicted for a rotor system with an unbalance load $m_u = 0.001 kg$ and a radius of unbalance $r_u = 0.025 m$. If the frequency of revolution $\Omega$ gets close to the frequency $\Omega_a$, the relative displacements between rotor and shaft is infinite. The same effect is known from linear rotor systems with unbalance load running in resonance, for the case, that no outer damping acts. In this respect the frequency $\Omega_a$ can be considered as a resonance frequency of the nonlinear system. The stability functions $s_1$, $s_2$ and $s_3$ are positive up to this frequency (figure 4).
Figure 4: Relative displacement $\Delta r_0$ between rotor and shaft with unbalance load and stability functions $s_1$, $s_2$ and $s_3$. $\Omega_1 = 542.2 \text{rad/s}$, $\Omega_a = 716.5 \text{rad/s}$. Parameters for calculation are $m_w = 0.192 \text{kg}$, $m_R = 4 \text{kg}$, $k_w = 2.15 \cdot 10^6 \text{N/m}$, $k_i = 2.68 \cdot 10^6 \text{N/m}$, $k_{int} = 2.01 \cdot 10^{16} \text{N/m}^3$, $c_i = 1.54 \cdot 10^5 \text{Ns/m}$, $m_u = 0.001 \text{kg}$, $r_u = 0.025 \text{m}$.

3 Comparison of the analytical solution with measurements and numerical results

In this section the analytically identified stability bounds and rotor deflections are compared with measurements and simulation results. For principal investigations a rotor system is applicable where two o-rings connect rotor and shaft. As the clearance between shaft and rotor is limited the restoring forces within the o-rings become nonlinear. This relation can be described comparatively well with a cubic force law like in equation (1). The more the relative displacements between rotor and shaft gets larger the more the o-ring stiffens. If the clearance between rotor and shaft is reached the joint stiffness is very high. This effect is best describable by a bilinear cubic force law, which is an enhancement of the cubic force law. It will be deduced in the following from separate tests.

By means of a simulation of the rotor system with two embedded o-rings the differences between the force laws can be shown.

3.1 Nonlinear force law of an o-ring within a rotor shaft joint

In order to get a force law which describes the properties of an o-ring embedded between shaft and rotor the test configuration shown in figure 5 was build. It was analyzed in a hydro-puls unit. The rotor was charged with

Figure 5: Test configuration with two o-rings analyzed in a hydro-pulse unit.

Figure 6: Hysteresis loop of the test configuration with two o-rings, simulation and measurement.

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a harmonic force. The frequency was 1.0 Hz and the amplitude of the force was 23.75 N. The displacement between rotor and shaft was measured by a capacitive sensor. The results are depicted in figure 6. The measured hysteresis loop flattens for large deflections as the clearance between rotor and shaft is reached. In addition a simulation of the test was performed numerically. The simulation enables to deduce appropriate parameters of different force laws.

The use of a linear force law for the o-ring (21) leads to an unacceptable difference between measurement and simulation.

\[ F_o = c_i \cdot \Delta \dot{x} + k_i \cdot \Delta x \]  

(21)

The linear damping coefficient of the o-ring is \( c_i \), \( k_i \) is the stiffness of the o-ring and \( \Delta x \) is the relative displacement between shaft and rotor.

A cubic o-ring model reflects the stiffening of the o-ring at high forces (22). The force deflection function remains smooth, such that the numerical implementation does not provoke any difficulties. Furthermore this force law implemented in a model of a rotor system enables the use of the stability bounds of section 2. A cubic stiffness \( k_{int} \) is used.

\[ F_o = c_i \cdot \Delta \dot{x} + k_i \cdot \Delta x + k_{int} \cdot \Delta x^3 \]  

(22)

However, this model has some drawbacks. The linear stiffness is too large and the bending of the curve is not reflected very well.

For a numerical simulation of the rotor system, a bilinear cubic force law can be used (23).

\[ F_o = \begin{cases} c_i \cdot \Delta \dot{x} + k_i \cdot \Delta x + k_{int} \cdot \Delta x^3 & : \Delta x \leq s \\ c_i \cdot \Delta \dot{x} + k_i \cdot \Delta x + k_{ir} \cdot (\Delta x - s) + k_{int} \cdot \Delta x^3 & : \Delta x > s \end{cases} \]  

(23)

A new linear relative stiffness \( k_{ir} \) is introduced which is multiplied by the difference between relative displacement \( \Delta x \) and clearance \( s \), and which acts only if the relative displacement is larger than the clearance. The linear relative stiffness \( k_{ir} \) is much larger than the linear stiffness. The simulated and measured hysteresis loops coincide well.

### 3.2 Results of measurement and simulation: rotor system with an o-ring embedded between shaft and rotor

Measurements are carried out on a rotor test rig, whose rotor shaft joint corresponds to the joint which was analyzed in the hydro puls unit in section 3.1. It is shown in figure 7. To avoid the weight influencing shaft bending and displacement between rotor and shaft the rotor system is arranged vertically. As the o-rings are largely compressed within the connection the weight of the rotor is absorbed by the o-rings. Nevertheless a plastic bending and displacement between rotor and shaft the rotor system is arranged vertically. As the o-rings are lubricated. No additional outer dampers are used. Thus a preferably basic rig design could be achieved allowing a more accurate simulation model.

The simulation model in figure 8 was generated using the software tool Feran [3]. The shaft is discretized by eight beam elements. The rotor is considered to be rigid. Three rotor nodes were created, in which two are connected with the shaft by a nonlinear force law. This model permits to compare the three different force laws of the rotor shaft joint. In addition an analytical calculation with equations of section 2 is performed. Parameters of the test rig are: mass of rotor \( m_R = 3.98 \text{ kg} \), shaft radius \( r = 8 \text{ mm} \), shaft length \( l = 247 \text{ mm} \) and clearance between shaft and rotor \( s = 0.011 \text{ mm} \). For simulation of one o-ring the following values are used: damping value \( c_i = 7.7 \cdot 10^4 \text{ Ns/m} \), linear joint stiffness (linear and cubic model) \( k_i = 1.565 \cdot 10^6 \text{ N/m} \), linear joint stiffness (bilinear cubic model) \( k_i = 1.34 \cdot 10^6 \text{ N/m} \), cubic stiffness (cubic model) \( k_{int} = 1.17375 \cdot 10^{16} \text{ N/m} \), cubic stiffness (bilinear cubic model) \( k_{int} = 1.608 \cdot 10^{15} \text{ N/m}^3 \), relative stiffness (bilinear cubic model) \( k_{ir} = 4.02 \cdot 10^7 \text{ N/m} \).

If one assume a cubic force law (22) the following characteristic frequencies are obtained from equations (14) and (15):

- Stability margin of the linear system \( f_1 = \frac{\Omega}{2\pi} = 89.3 \text{ Hz (}= 5358 \text{ U/min}) \)
- Stability margin of the nonlinear system \( f_a = \frac{\Omega}{2\pi} = 114.3 \text{ Hz (}= 6858 \text{ U/min}) \).
In figure 9 simulation results of a run up are shown for different force laws of the o-ring. The simulation was performed with an unbalance load of 30 g and a radius of unbalance of 25 mm. The frequency of revolution was altered linearly to 120 Hz within a period of 6 s.

It is obvious, that a stabilizing effect beyond the linear stability margin is only obtained if a nonlinear model is used for the o-ring. The simulated rotor displacements for a cubic and bilinear cubic model differ only marginally. Furthermore it becomes clear that the analytically deduced stability margin of the nonlinear system can be used to estimate the stability margin of the system simulated with a bilinear cubic force law. These results will be validated on the basis of measurements.

**3.2.1 Measurement results in comparison with simulation results**

As the o-rings are the sole nonlinear elements with internal damping on this rotor system, the test is especially applicable to validate the simulation model of the rotor system with bilinear cubic force law. Multiple measurements were carried out on this test rig. With the help of two laser triangulation sensors the deflection of shaft and rotor were measured. The results are depicted in figure 10. Figure 7 shows the positions of both sensors.

Within the simulation the bilinear cubic force law of equation (23) is used to describe the properties of the two
o-rings. The unbalance load of the test rotor is unknown. For calculation the unbalance value is adjusted to the measured amplitudes. This yields an unbalance load of $m_u = 30 \, g$ with a radius of unbalance $r_u = 25 \, mm$.

The curve obtained by simulation coincide qualitatively with the measurements (figure 10). Especially on comparing measurements with analytical and numerical results (figure 9) one can assert that the stability margin of the nonlinear system can be estimated by an analytical calculation and that the stability margin is moved up towards higher frequencies. Furthermore the test shows that the implemented o ring model is appropriate for the description of the rotor dynamics of this rotor shaft joint.

A measurement beyond of the stability margin is not possible on this test rig. Above $95 \, Hz$ the run of the test rig becomes very noisy. For safety reasons measurements were ceased at a frequency of revolution of $100 \, Hz$.

4 Conclusion

A four degrees of freedom model of a rotor system whose rotor shaft joint is subject of a nonlinear force law was stated for calculating rotor deflections and stability margins analytically. It could be shown that the nonlinearity enlarges the stability range compared two a linear rotor shaft joint. The influence of several parameters to the stability margin were worked out.

The analytically deduced relations were confirmed by numerical simulations and measurements on a rotor test rig. The rotor shaft joint was designed with two o-rings possessing nonlinear characteristics within the joint. Parameters and a suitable force law of this joint were deduced in a separate test. A simulation of the whole rotor system with the derived force law coincides well with measurement results. It turned out to be possible to estimate the stability margin of the test rig via the analytically calculated boundary frequency.

As in the test rig design no additional outer dampers were used a supercritical operation of the rotor system was not possible. This should be a task for further research.

References


